Arithmetical Foundations
Recursion. Evaluation. Consistency

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Preface

Recursive maps, nowadays called *primitive recursive maps,* p. r. maps, have been introduced by GÖDEL in his 1931 article for the arithmetisation, *gödelisation,* of metamathematics.

For construction of his *undecidable formula* he introduces a non-constructive, non-recursive predicate *beweisbar,* prov*able.*

Staying within the area of (categorical) free-variables theory **PR** of primitive recursion or appropriate extensions opens the chance to avoid the two Gödel's incompleteness theorems: these are stated for *Principia Mathematica und verwandte Systeme,* "related systems" such as in particular Zermelo-Fraenkel set theory **ZF** and v. Neumann Gödel Bernays set theory **NGB**.

On the basis of primitive recursion we consider μ-recursive maps as *partial p. r. maps.* Special *terminating* general recursive maps considered are *complexity controlled* iterations. *Map code evaluation* for **PR** is given in terms of such an iteration.

We discuss iterative map code evaluation in direction of *termination conditioned soundness,* and based on this μ-recursive decision of primitive recursive predicates. This leads to consistency provability and soundness for classical, quantified arithmetical and set theories as well as for the p. r. *descent* theory $\pi\mathbf{R}$, with unexpected consequences:

We show *inconsistency provability* for the quantified theories, as well as *consistency provability* and logical *soundness* for the theory $\pi\mathbf{R}$ of primitive recursion, strengthened by an axiom scheme of *non-infinite descent of complexity controlled iterations* like (iterative) map-code evaluation.

Berlin, April 2014 M. Pfender.

Contents

Introduction

We fix *constructive foundations* for arithmetic on a *map* theoretical, *algorithmical* level. In contrast to *elementhood* and *quantification* based traditional foundations such as Principia Mathematica **PM** or Zermelo-Fraenkel set theory **ZF**, our *fundamental primitive recursive theory* **PR** has as its "undefined" terms just terms for objects and maps. On that language level it is *variable free,* and it is free from formal quantification on individuals like numbers or number pairs.

This theory **PR** is a formal, *combinatorial category* with cartesian i. e. universal *product* and a natural numbers object (NNO) $\mathbb{N}$, a *p. r. cartesian category,* cf. Romàn 1989.

The NNO $\mathbb{N}$ admits *iteration of endo maps* and the *full scheme of primitive recursion.* Such NNO has been introduced in categorical terms by Freyd 1972, on the basis of the NNO of Lawvere 1964.

We will remain on the purely *syntactical* level of this categorical theory, and later extensions: *no formal semantics* necessary into an outside, non-combinatorial world. Cf. Hilbert's formalistic program.

We then introduce into our *variable-free* setting *free variables,* as *names* for identities and projections. As a consequence, we have in the present context '*free variable*' as a *defined* notion. We have object and map constants such as terminal object, NNO, zero etc. and use free metavariables for objects and for maps.

Fundamental arithmetic is further developed along GOODSTEIN's 1971 *free variables Arithmetic* whose *uniqueness rules* are derived as theorems of categorical theory **PR**, with its "eliminable" notion of a *free variable.* This gives the expected structure theorem for the algebra and order on NNO $\mathbb{N}$. "On the way", via Goodstein's *truncated subtraction,* and his commutativity of maximum function, we obtain the *Equality Definability theorem:* If predicative equality of two p. r. maps is derivably true, then map equality between these maps is derivable. It follows a subsection on the derivation of the Peano axioms as theorems.

The subsequent chapter brings into the game an embedding theory extension of **PR** by *abstraction* of *predicates* into "virtual" new *objects.* This enrichment makes emerging *basic* theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ more comfortable, in direction to set theories, with their *sets* and *subsets.*

chapter 3 introduces the general concept of *partial* maps, proves a structure theorem on the theory $\mathbf{P\widehat{R}a}$ of these maps and shows that μ-recursive maps and while-loop programs are just partial p. r. maps; in particular our evaluations will be such (formally) partial maps.

Partial maps are introduced here as map pairs consisting of a *domain-of-definition enumeration* (in general not mono) and of a *rule* to throw an enumeration index of a *defined argument* into the *value* of that argument. *Equality* of partial maps is by availability of extension maps between the enumeration domains of the two partial maps under consideration, in both directions.

These partial maps form a primitive recursive diagonal-monoidal half-cartesian theory $\mathbf{P\widehat{R}a}$ (cf. Budach & Hoehncke 1975) which contains theory $\mathbf{PRa}$ embedded as theory of this type, composition being defined via composition of pullbacks: Structure theorem for partials. Theory extension by partiality is a *closure* operation: *partial* partial maps are just partial maps.

chapter 4 then exhibits within theory $\mathbf{PRa}$ a *universal object* $\mathbb{X}$, of all *numerals* and nested pairs of numerals, and constructs by means of that object *universe theories* $\mathbf{PR}\mathbb{X}$ and $\mathbf{PR}\mathbb{X}\mathbf{a}$: theory $\mathbf{PR}\mathbb{X}$ is good for a one-object map-code evaluation, $\mathbf{PR}\mathbb{X}\mathbf{a}$ contains $\mathbf{PRa}$ as a cartesian p. r. embedded theory with predicate extensions.

chapter 5 on *evaluation* strengthens p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ into *descent theory* $\pi\mathbf{R}$, by an axiom of *non-infinite iterative descent* with order values in polynomial semiring $\mathbb{N}[\omega]$ ordered (reverse-)lexicographically.

This theory is shown to derive the—free variable p. r.—consistency formula for theories $\mathbf{PR}\mathbb{X}\mathbf{a}$ (and $\mathbf{PR}$). The proof relies on constructive, *complexity controlled* code evaluation, which is extended to evaluation of *argumented deduction trees:*

Theorem on *p. r. soundness* within **set** theory as frame (chapter 6), and *termination conditioned soundness* of $\mathbf{PRa} \subset \mathbf{PR}\mathbb{X}\mathbf{a}$ within theory $\pi\mathbf{R}$ taken as frame (chapter 7).

The consequence is decidability of p. r. predicates within both theories. Since consistency formulae Con of both theories can be expressed as (free variable) p. r. predicates, this leads to

1. *Inconsistency provability* of **set** theory by Gödel's second incompleteness theorem, and to

2. *Consistency provability* and soundness of descent theory $\pi\mathbf{R}$,

under ***assumption*** of μ-consistency.

The latter is a (**set** theoretically) equivalent variant of ω-consistency.

Chapter 1

Primitive Recursion

Almost everything in this long first chapter is known from classical free-variables Arithmetic, see GOODSTEIN 1971. What we need formally for our categorical p. r. free-variables Arithmetic is categorical schemes of iteration and primitive recursion, categorical interpretation of free variables as identities and projections out of cartesian products as well as proof of GOODSTEIN's rules U_1-U_4 for the categorical theory **PR** of primitive recursion developped here from scratch.

1.1 Fundamental theory PR of primitive recursion

We fix here terms and axioms for the *fundamental* categorical (formally variable-free) cartesian theory **PR** of primitive recursion.

The basic objects of the theory **PR** are the *natural numbers object* ('NNO') $\mathbb{N}$ and the *terminal* object $\mathbb{1}$.

Composed objects of **PR** come in as *"cartesian" products* $(A \times B)$ of objects already enumerated. Formally:

$$(\mathrm{Obj}_{\mathrm{Cart}}) \quad \frac{A, B \text{ objects}}{(A \times B) \text{ object}}$$

[Here outmost brackets may be dropped]

Maps: *Basic maps* ("map constants") of the theory **PR** are

the *zero map* $0 : \mathbb{1} \to \mathbb{N}$, and

the *successor map* $s : \mathbb{N} \to \mathbb{N}$

Structure of PR as a category:

- generation—enumeration—of *identity maps*

$$(\text{id generation}) \quad \frac{A \text{ an object}}{\mathrm{id}_A : A \to A \text{ map}}$$

- Composition:

$$(\circ) \quad \frac{f : A \to B,\ g : B \to C \text{ maps}}{(g \circ f) : A \to C \text{ map, diagram:}}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Here are the axioms making **PR** into a category:

- **Associativity** of *composition:*

$$(\circ_{\mathrm{ass}}) \quad \frac{f : A \to B,\ g : B \to C,\ h : C \to D \text{ maps}}{h \circ (g \circ f) = (h \circ g) \circ f : A \to D}$$

- **Neutrality** of *identities*

$$(\mathrm{neutr}_{\mathrm{id}}) \quad \frac{f : A \to B \text{ map}}{\begin{array}{l}(f \circ \mathrm{id}_A) = f : A \to A \to B \quad \text{and} \\ (\mathrm{id}_B \circ f) = f : A \to B \to B.\end{array}}$$

map equality $f = g : A \to B$ satisfies the axioms of reflexivity, symmetry, and transitivity:

$$(\mathrm{refl}) \quad \frac{f : A \to B \text{ map}}{f = f : A \to B}$$

$$(\mathrm{sym}) \quad \frac{f = g : A \to B \text{ map}}{g = f : A \to B}$$

$$\text{(trans)} \quad \frac{f = g,\ g = h : A \to B \text{ maps}}{f = h : A \to B}$$

Composition is compatible with equality:

$$(\circ_=) \quad \frac{f = f' : A \to B,\ g = g' : B \to C}{(g \circ f) = (g' \circ f') : A \to B \to C}$$

Because of technical simplicity in later code evaluation, we split this axiom into the following two ones:

$$(\circ_= \text{1st}) \quad \frac{f = f' : A \to B,\ g : B \to C}{(g \circ f) = (g \circ f') : A \to B \to C}$$

$$(\circ_= \text{2nd}) \quad \frac{f : A \to B,\ g = g' : B \to C}{(g \circ f) = (g' \circ f) : A \to B \to C}$$

Cartesian map structure:

- enumeration of *terminal maps*

$$\frac{A \text{ object}}{\Pi = \Pi_A : A \to \mathbb{1} \text{ map}}$$

[In Eilenberg & Elgot's notation. Lawvere designates this projection $! : A \to \mathbb{1}$.]

- uniqueness axiom for terminal map family:

$$(\Pi) \quad \frac{A \text{ object},\ f : A \to \mathbb{1} \text{ map}}{f = \ \Pi_A : A \to \mathbb{1}}$$

Π-naturality **Lemma:** $\Pi = [\Pi : A \to \mathbb{1}]_A$ is natural, i. e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Pi_A"'] \arrow[dr, phantom, "="] & B \arrow[d, "\Pi_B"] \\ \mathbb{1} \arrow[r, equal, "\mathrm{id}"] & \mathbb{1} \end{tikzcd}$$

- generation of left and right *projections:*

$$(\text{proj}) \quad \frac{A,\ B \text{ objects}}{\begin{array}{l} l = l_{A,B} : A \times B \to A \ \textit{left projection,} \\ r = r_{A,B} : A \times B \to B \ \textit{right projection} \end{array}}$$

- generation of *induced maps* into products:

$$\text{(ind)} \quad \frac{f : C \to A,\ \ g : C \to B \text{ maps}}{\begin{array}{l}(f,g) : C \to A \times B \text{ map,} \\ \text{the map } \textit{induced} \text{ by } f \text{ and } g\end{array}}$$

- compatibility of induced map formation with equality:

$$(\text{ind}_=) \quad \frac{f = f' : C \to A,\ g = g' : C \to B \quad \text{maps}}{(f,g) = (f',g') : C \to A \times B}$$

- characteristic (Godement) equations

$$(\text{GODE}_l) \quad \frac{f : C \to A,\ g : C \to B}{l \circ (f,g) = f : C \to A}$$

as well as

$$(\text{GODE}_r) \quad \frac{f : C \to A,\ g : C \to B}{r \circ (f,g) = g : C \to B}$$

in *commutative* diagram form:

$$\begin{tikzcd}
 & & A \\
C \arrow[urr, bend left, "f"] \arrow[rr, "{(f,g)}"] \arrow[drr, bend right, "g"'] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & & A \times B \arrow[u, "l"'] \arrow[d, "r"] \\
 & & B
\end{tikzcd}$$

- uniqueness of induced map (Godement):

$$\text{(ind!)} \quad \frac{\begin{array}{l} f : C \to A,\ g : C \to B,\ h : C \to A \times B \text{ maps}, \\ l \circ h = f : C \to A \text{ and } r \circ h = g : C \to B \end{array}}{h = (f, g) : C \to A \times B}$$

SP Lemma: In presence of the other axioms, this *uniqueness of the induced map* is equivalent to the following equational axiom of *Surjective Pairing*, see Lambek-Scott 1986:

$$\text{(SP)} \quad \frac{h : C \to A \times B}{(l \circ h\, ,\, r \circ h) = h : C \to A \times B}$$

Proof as an **exercise:** Use compatibility of forming the induced map with equality.

We will formally rely on this equation as an axiom. It replaces uniqueness of forming the induced map.

We eventually replace equivalently, given the other axioms, inferential axiom ($\mathrm{ind}_=$) by *distributivity equation*

$$(\mathrm{distr}_\circ)\quad \frac{h : D \to C,\ f : C \to A,\ g : C \to B}{(f, g) \circ h = (f \circ h, g \circ h) : D \to A \times B}$$

taken from Lambek-Scott. Equivalence proof as an **exercise,** proof of *uniqueness of the induced* in op. cit. Draw the diagram.

Definition: we define, for a map $g : B \to B'$, *cylindrification*

$$A \times g \ =_{\mathrm{def}}\ \mathrm{id}_A \times g \ =_{\mathrm{def}}\ (\mathrm{id}_A \circ l, g \circ r) : A \times B \to A \times B'.$$

Diagram:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] & A \\
A \times B \arrow[u, "l"] \arrow[d, "r"'] \arrow[r, dotted, "A \times g"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & A \times B' \arrow[u, "l"'] \arrow[d, "r"] \\
B \arrow[r, "g"] & B'
\end{tikzcd}$$

This ends the list of axioms for the *cartesian structure* of the theory **PR**.

Axioms for the iteration of endo maps:

$$(\S)\quad \frac{f : A \to A \text{ (endo) map}}{\begin{array}{l} f^\S : A \times \mathbb{N} \to A \textit{ iterated of } f \text{, satisfies} \\ f^\S \circ (\mathrm{id}_A, 0) = \mathrm{id}_A : A \to A \quad [0 := 0\,\Pi] \text{ (anchor)}, \\ f^\S \circ (A \times s) = f \circ f^\S : A \times \mathbb{N} \to A \to A \text{ (step)}. \end{array}}$$

In ***free-variables*** notation (see below):

$$\begin{array}{lr} f^\S(a, 0) = a & \text{(anchor)}, \\ f^\S(a, s\,n) = f(f^n(a)) \ =_{\text{by def}} \ f(f^\S(a, n)) & \text{(step)}. \end{array}$$

In commuting-diagram form,

“Pentagonal” diagram:

$$\begin{tikzcd}
& A \times \mathbb{N} \arrow[r, "A \times s"] \arrow[dd, "f^\S"] & A \times \mathbb{N} \arrow[dd, "f^\S"] \\
A \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
& A \arrow[r, "f"] & A
\end{tikzcd} \qquad \text{(it)}$$

basic iteration DIAGRAM

As a first **example** for an iterated endo map take *addition*

$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, having properties

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[d, "+"', "s^{\S}"] \arrow[dr, phantom, "="] & A \times \mathbb{N} \arrow[d, "+"', "s^{\S}"] \\
A \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & A \arrow[r, "s"] & A
\end{tikzcd} \qquad (+)$$

i. e. satisfying the free-variables equations

$$a + 0 = a : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$a + s\,n = s(a+n) = (a+n) + 1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{s} \mathbb{N},$$
$$\text{where } 1 \ =_{\text{def}}\ s \circ 0 : \mathbb{1} \to \mathbb{N} \to \mathbb{N}.$$

[A formal introduction of free variables as projections see below.]

uniqueness axiom for the iterated:

$$(\S!) \quad \frac{\begin{array}{l} f : A \to A \text{ (endo map)} \\ h : A \times \mathbb{N} \to A, \\ h \circ (\mathrm{id}_A, 0) = \mathrm{id}_A \text{ and} \\ h \circ (A \times s) = f \circ h \text{ “as well”} \end{array}}{h = f^{\S} : A \times \mathbb{N} \to A}$$

By this uniqueness axiom, the iterated map is characterised by the commutative pentagonal diagram above.

Theorem (compatibility of iteration with equality): uniqueness axiom (§!) infers

$$(\S_=) \quad \frac{f = g : A \to A}{f^\S = g^\S : A \times \mathbb{N} \to A}$$

Proof: Consider the diagram

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A \times s"] \arrow[dd, dotted, bend right, "f^\S"'] \arrow[dd, bend left, "g^\S"] \arrow[dd, phantom, "?"] \arrow[ddr, phantom, "="] & A \times \mathbb{N} \arrow[dd, dotted, bend right, "f^\S"'] \arrow[dd, bend left, "g^\S"] \arrow[dd, phantom, "?"] \\
A \arrow[ur, "{(\mathrm{id},0)}"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & & \\
 & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "="] & A
\end{tikzcd}$$

Since $f^\S$ is the *unique* commutative fill-in into this pentagonal diagram over endomorphism f, it is sufficient to show that $g^\S : A \times \mathbb{N} \to A$ equally is such a commutative fill in.

For the triangle (anchor) this is trivial: $g^\S(\mathrm{id}, 0) = \mathrm{id} : A \to A$ by definition of the null-fold iterated.

For the square (step) we have

$$\begin{aligned}
&g^\S \circ (A \times s) = g \circ g^\S \ (\textit{definition of } g^\S) \\
&= f \circ g^\S : A \times \mathbb{N} \to A,
\end{aligned}$$

by assumption $f = g$ and by compatibility of $\circ$ with $=$ in first composition factor, axiom ($\circ_=$1st).

So $g^{\S}$ turns out to be another iterated of endo f, whence in fact $g^{\S} = f^{\S}$ by uniqueness of the iterated **q. e. d.**

These axioms give all objects and maps of theory **PR**.

Freyd's uniqueness scheme which completes the axioms constituting theory **PR**, reads in free variables notation:

$$\frac{\begin{array}{l} f : A \to B \text{ (init map)}, \ g : B \to B \text{ (endo to be iterated)} \\ h : A \times \mathbb{N} \to B \text{ comparison map:} \\ h(a, 0) = f(a) \text{ (init)} \\ h(a, s\,n) = g(h(a, n)), \text{ (step)} \end{array}}{h(a, n) = g^n(f(a)) \text{ (uniqueness)},}$$

without use of free variables:

$$(\mathrm{FR!}) \quad \frac{\begin{array}{l} f : A \to B, \ g : B \to B, \, h : A \times \mathbb{N} \to B, \\ h \circ (\mathrm{id}_A, 0 \circ \Pi_A) = f : A \to B, \text{ (init)} \\ h \circ (A \times s) = g \circ h : A \times \mathbb{N} \to B, \text{ (step)} \end{array}}{h = g^{\S} \circ (f \times \mathbb{N}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B,}$$

in form of Freyd's pentagonal diagram:

$$
\begin{tikzcd}
 & & A \times \mathbb{N} \arrow[rr, "A\times s"] \arrow[d, "f\times\mathbb{N}"'] \arrow[dd, dotted, bend left, "h"] & & A \times \mathbb{N} \arrow[d, "f\times\mathbb{N}"'] \arrow[dd, dotted, bend left, "h"] \\
A \arrow[urr, "{(\mathrm{id},0)}"] \arrow[drr, "f"'] & = & B \times \mathbb{N} \arrow[d, "g^{\S}"'] & = & B \times \mathbb{N} \arrow[d, "g^{\S}"'] \\
 & & B \arrow[rr, "g"] & & B
\end{tikzcd}
$$

FREYD's uniqueness DIAGRAM (FR!)

Remark: This uniqueness of the *initialised iterated* obviously specialises to axiom (§!) of uniqueness of "simple" iterated $f^{\S} : A \times \mathbb{N} \to A$ and so makes that uniqueness axiom redundant.

Problem: Is, conversely, stronger Freyd's uniqueness axiom already covered by uniqueness (§!) of "simply" iterated $f^{\S} : A \times \mathbb{N} \to A$? My guess is "no".

Freyd's *existence* and *uniqueness* of the *initialised iterated* is displayed as the following commutative diagram:

$$
\begin{tikzcd}
 & & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[d, "f\times\mathbb{N}"'] \arrow[dr, "f\times s"] \arrow[ddd, dotted, bend left, "h"] & A \times \mathbb{N} \arrow[d, "f\times\mathbb{N}"] \arrow[ddd, dotted, bend left, "h"] \\
A \arrow[urr, "(\mathrm{id},0)"] \arrow[dr, "f"'] \arrow[rr, phantom, "="] & & B \times \mathbb{N} \arrow[r, "A\times s"] \arrow[dd, "g^{\S}"'] \arrow[ddr, phantom, "="] & B \times \mathbb{N} \arrow[dd, "g^{\S}"] \\
 & B \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & & \\
 & & B \arrow[r, "g"] & B
\end{tikzcd}
$$

FREYD's uniqueness DIAGRAM (FR!)

Proof: Existence of $g^{\S}$ and commutativity of lower triangle and square follow directly from axiom (§). Upper right commutativity is splitting a cartesian product $f \times s$ in the two ways into compositions of right and left cylindrified maps.

Remaining equation

$$(\mathrm{id}_B, 0 \circ \Pi_B) \circ f = (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0 \circ \Pi_A) : A \to B \times \mathbb{N}$$

is given by uniqueness of the induced map into the cartesian product $B \times \mathbb{N}$, in detail:

$$
\begin{aligned}
l \circ (\mathrm{id}_B, 0) \circ f = \mathrm{id}_B \circ f = f \quad \text{and} \\
l \circ (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0) = f \circ l \circ (\mathrm{id}_A, 0) = f \circ \mathrm{id}_A = f, \\
r \circ (\mathrm{id}_B, 0) \circ f = 0 \circ f = 0 \circ \Pi_A \quad \text{and} \\
r \circ (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0 \circ \Pi_A) = r \circ (\mathrm{id}_A, 0) = 0 \circ \Pi_A.
\end{aligned}
$$

Together this shows constructive *availability* of wanted *initialised iterated* $h : A \times \mathbb{N} \to B$.

Uniqueness of h, namely

$$(\mathrm{FR!}) \quad \frac{\begin{array}{l} f : A \to B,\ g : B \to B,\ h : A \times \mathbb{N} \to B \\ h \circ (\mathrm{id}_A, 0) = f \\ h \circ (A \times s) = g \circ h \end{array}}{h = g^{\S} \circ (f \times \mathbb{N}).}$$

above, is just required as an axiom, final axiom of theory **PR**.

From (FR!) we get trivially, with data

$$A \xrightarrow{\mathrm{id}_A} A \xrightarrow{f} A \ \text{specializing data}\ A \xrightarrow{f} B \xrightarrow{g} B$$

uniqueness (§!) of *iterated* map $f^{\S} : A \times \mathbb{N} \to A$.

1.2 The full scheme of primitive recursion

Already for definition and characterisation of *multiplication* and moreover for proof of "the" laws of arithmetic, the following *full scheme* (pr) of primitive recursion is needed:[1]

Theorem (Full scheme of primitive rec.): PR admits scheme

[1] in pure categorical form see FREYD 1972, and (then) PFENDER, KRÖPLIN, and PAPE 1994, not to forget its uniqueness clause

$$
\text{(pr)} \quad \frac{\begin{array}{l} g = g(a) : A \to B \ \ (\textit{init map}) \\ h = h(a, n) : (A \times \mathbb{N}) \times B \to B \ \ (\textit{step map}) \end{array}}{\begin{array}{l} f = f(a, n) : A \times \mathbb{N} \to B \\ \text{is given such that} \\ f(a, 0) = g(a) \text{ and} \\ f(a, s\,n) = h((a, n), f(a, n)) \\ \text{as well as} \\ (\text{pr!}) : f \text{ is } \textit{unique} \text{ with these properties.} \end{array}}
$$

Same without use of free variables:

$$
\text{(pr)} \quad \frac{\begin{array}{l} g : A \to B, \\ h : (A \times \mathbb{N}) \times B \to B \end{array}}{\begin{array}{l} \mathrm{pr}[g, h] := f : A \times \mathbb{N} \to B, \\ f(\mathrm{id}_A, 0) = g : A \to B, \\ f\,(\mathrm{id}_A \times s) = h\,(\mathrm{id}_{A \times \mathbb{N}}, f) : \\ (A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B, \\ (\text{pr!}) : f \ \textit{unique.} \end{array}}
$$

This scheme is an axiom in the classical theory of primitive recursion.

The categorical proof below out of existence and uniqueness of the initialised iterated may be delayed for categorical study a posteriori.

Proof of schema (pr) :

Construction of the map $f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$ out of data $g : A \to B$ (*initialisation*) and $h : (A \times \mathbb{N}) \times B \to B$ (*iteration step*):

Wanted $f : A \times \mathbb{N} \to B$ is to satisfy (init) und (step) given as the two commuting DIAGRAMS

$$
\begin{tikzcd}
 & A \times \mathbb{N} \arrow[d, "f"] \\
A \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "g"'] \arrow[r, phantom, "="] & {} \\
 & B
\end{tikzcd}
$$

(init)

$$
\begin{tikzcd}
(a,n) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & (a, s\,n) \arrow[dd, mapsto] \\
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[d, "(\mathrm{id},f)"'] \arrow[dr, phantom, "="] & A \times \mathbb{N} \arrow[d, "f"] & \\
 & (A \times \mathbb{N}) \times B \arrow[r, "h"] & B & f(a, s\,n) \arrow[dl, equal] \\
((a,n), f(a,n)) \arrow[rr, mapsto] & & h((a,n), f(a,n)) &
\end{tikzcd}
$$

(step)

With $\hat{g} := ((\mathrm{id}_A, 0), g)$ and $\hat{h} := ((A \times s) \circ l, h)$ we get by (FR!) a uniquely determined map

$$k = (k_l, k_r) : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$$

satisfying

$$\begin{tikzcd}
& A\times\mathbb{N} \arrow[rr, "A\times s"] \arrow[dd, dotted, "k"', "(k_l{,}k_r)"] & & A\times\mathbb{N} \arrow[dd, dotted, "k"', "(k_l{,}k_r)"] \\
A \arrow[ur, "(\mathrm{id}{,}0)"] \arrow[dr, "\hat{g}"'] \arrow[r, phantom, "="] & & = & \\
& (A\times\mathbb{N})\times B \arrow[rr, "\hat{h}", "((A\times s)\circ l{,}h)"'] & & (A\times\mathbb{N})\times B
\end{tikzcd}$$

i. e.

$$\begin{aligned}
&k \circ (\mathrm{id}_A, 0) = \hat{g} \quad \text{and} \\
&k \circ (A \times s) = \hat{h} \circ k.
\end{aligned}$$

[It will turn out that $k = (\mathrm{id}_{A\times\mathbb{N}}, f)$ for wanted map $f : A \times \mathbb{N} \to B$.]

For our unique k, consider first its left component $k_l = l \circ k : A \times \mathbb{N} \to A \times \mathbb{N}$, unique—by (FR!)—in

$$\begin{tikzcd}
& A\times\mathbb{N} \arrow[rr, "A\times s"] \arrow[dd, "k"'] \arrow[ddd, dotted, bend left, "k_\ell"', "\mathrm{id}"] & & A\times\mathbb{N} \arrow[dd, "k"'] \arrow[ddd, dotted, bend left, "k_l"', "\mathrm{id}"] \\
A \arrow[ur, "(\mathrm{id}{,}0)"] \arrow[dr, "\hat{g}"'] \arrow[ddr, bend right, "(\mathrm{id}{,}0)"'] \arrow[r, phantom, "="] & & = & \\
& (A\times\mathbb{N})\times B \arrow[rr, "\hat{h}", "((A\times s)\circ l{,}h)"'] \arrow[d, "l"'] & & (A\times\mathbb{N})\times B \arrow[d, "l"'] \\
& A\times\mathbb{N} \arrow[rr, "A\times s"] & & A\times\mathbb{N}
\end{tikzcd}$$

We have

$$\begin{aligned} l \circ k \circ (\mathrm{id}_A, 0) &= l \circ \hat{g} = (\mathrm{id}_A, 0) \quad \text{and} \\ l \circ k \circ (A \times s) &= l \circ \hat{h} \circ k = (A \times s) \circ l \circ k \end{aligned}$$

Since these two equations hold likewise for $\mathrm{id}_{A\times\mathbb{N}}$ instead of $l \circ k$, it follows by uniqueness (FR!) of such a map $l \circ k = \mathrm{id}_{A\times\mathbb{N}}$.

Taking now $f := r \circ k : A \times \mathbb{N} \to B$, we have the following diagram for this (unique) right component of $k : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$:

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[rr, "A\times s"] \arrow[d, "k"'] \arrow[dd, bend left, "f"] \arrow[drr, phantom, "="] & & A \times \mathbb{N} \arrow[d, "k"'] \arrow[dd, bend left, "f"] \\
A \arrow[ur, "{(\mathrm{id},0)}"] \arrow[r, "\hat{g}", "{((\mathrm{id},0),g)}"'] \arrow[drr, phantom, "="] \arrow[dr, bend right, "g"'] \arrow[u, phantom, "="] & (A \times \mathbb{N}) \times B \arrow[rr, "\hat{h}", "{((A\times s)\circ l,h)}"'] \arrow[d, "r"'] \arrow[drr, "h"'] & & (A \times \mathbb{N}) \times B \arrow[d, "r"'] \\
 & B & & B
\end{tikzcd}$$

obtain

$$\begin{aligned} k &= (l \circ k, r \circ k) = (\mathrm{id}_{A\times\mathbb{N}}, f), \\ f \circ (\mathrm{id}_A, 0) &= r \circ k \circ (\mathrm{id}_A, 0) = r \circ \hat{g} = g \quad \text{and} \\ f \circ (A \times s) &= r \circ k \circ (A \times s) = r \circ \hat{h} \circ k \\ &= h \circ k = h \circ (\mathrm{id}_{A\times\mathbb{N}}, f) \end{aligned}$$

So this map $f : A \times \mathbb{N} \to B$ is *available,* to fullfill the requirements of $\mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$.

uniqueness proof for such map f: Let f' be a map assumed likewise to satisfy equations (init) and (step).

Then take $k' := (\mathrm{id}_{A\times\mathbb{N}}, f') : A\times\mathbb{N} \to (A\times\mathbb{N}) \to B$ and calculate:

$$\begin{aligned} k' \circ (\mathrm{id}_A, 0) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (\mathrm{id}_A, 0) \\ &= ((\mathrm{id}_A, 0), f' \circ (\mathrm{id}_A, 0)) \\ &= ((\mathrm{id}_A, 0), g) = \hat{g} \quad \text{as well as} \\ k' \circ (A \times s) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (A \times s) \\ &= ((A \times s), f' \circ (A \times s)) \\ &= ((A \times s), h) = \hat{h} \circ k'. \end{aligned}$$

Since by (FR!), k above is the *unique* map to satisfy the equations above, we have necessarily $k' = k$ and hence $f' = r \circ k' = r \circ k = f : A \times \mathbb{N} \to B$. **q. e. d.**

1.3 Uniqueness of the NNO $\mathbb{N}$

Category theorists like constructions which are *uniquely* given by their defining properties, unique up to *natural isomorphisms,* or—functorial constructions—up to natural equivalence. For the (binary) cartesian product with its projection families as *natural map* families this is true by considerations earlier above, same for the family $\Pi : A \to \mathbb{1}$ of terminal maps (projections.)

Now what about the natural numbers object

$$\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}\,?$$

This DIAGRAM has the property wanted, property which should be called *categoricity:* by its LAWVERE *existence* and *uniqueness* properties below, it is just the *initial diagram* $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ of form

$\mathbb{1} \xrightarrow{a_0} A \xrightarrow{f} A$.

So, *purely map theoretically,* the notion of an NNO *is categoric:* Within a cartesian map theory, NNO $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is unique, up to *natural isomorphism.*

Specialised, sequences definition of NNO: Lawvere defines the NNO $\mathbb{N}$ as follows:

$$(\mathrm{NNO}_{\mathrm{FWL}}) \quad \frac{\begin{array}{l} a_0 : \mathbb{1} \to A \text{ a } \textit{point,} \\ f : A \to A \text{ an endo map } \textit{to be iterated} \end{array}}{\begin{array}{l} a : \mathbb{N} \to A \textit{ resulting sequence,} \\ a \circ 0 = a_0 : \mathbb{1} \to A, \textit{ start of sequence,} \\ a \circ s = f \circ a : \mathbb{N} \to A \textit{ progress of sequence} \\ + \textit{ uniqueness} \text{ of such sequence } a : \mathbb{N} \to A, \end{array}}$$

in diagram form:

$$\begin{tikzcd}
& \mathbb{N} \arrow[r, "s"] \arrow[dd, dotted, "a"] \arrow[ddr, phantom, "="] & \mathbb{N} \arrow[dd, dotted, "a"] \\
\mathbb{1} \arrow[ur, "0"] \arrow[dr, "a_0"'] \arrow[r, phantom, "="] & {} & \\
& A \arrow[r, "f"] & A
\end{tikzcd}$$

Lawvere NNO diagram

We show that this early NNO scheme is obtained from Freyd's scheme.

NNO Lemma: For $a_0 : \mathbb{1} \to A$ and $f : A \to A$ (antecedent in LAWVERE's NNO scheme), the map

$$a \;=_{\mathrm{def}}\; f^\S \circ (a_0, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \to A \times \mathbb{N} \xrightarrow{f^\S} A$$

uniquely makes the above diagram commute.

Proof: Consider the following DIAGRAM:

$$\begin{tikzcd}
 & \mathbb{1}\times\mathbb{N} \arrow[r, "\mathbb{1}\times s"] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, phantom, "="] & \mathbb{1}\times\mathbb{N} \arrow[d, leftrightarrow, "\cong"] \\
 & \mathbb{N} \arrow[r, "s"] \arrow[d, "{(a_0,\mathrm{id})}"] \arrow[dd, dotted, bend right, "h"'] \arrow[dr, phantom, "="] & \mathbb{N} \arrow[d, "{(a_0,\mathrm{id})}"'] \arrow[dd, dotted, bend left, "h"] \\
\mathbb{1} \arrow[uur, "{(\mathrm{id},0)}"] \arrow[ur, "0"] \arrow[r, "{(a_0,0)}"] \arrow[dr, bend right, "a_0"'] & A\times\mathbb{N} \arrow[r, "A\times s"] \arrow[d, "f^\S"] \arrow[dr, phantom, "="] & A\times\mathbb{N} \arrow[d, "f^\S"'] \\
 & A \arrow[r, "f"] & A
\end{tikzcd}$$

FREYD to LAWVERE NNO specialisation DIAGRAM

Here $h = h(n) : \mathbb{N} \to A$ is another *sequence* assumed to fullfill the postcedent above in place of $a : \mathbb{N} \to A$: That

$$a \;=_{\text{by def}}\; f^\S \circ (a_0, \mathrm{id}_{\mathbb{N}}) \circ \cong : \mathbb{1} \times \mathbb{N} \to \mathbb{N} \to A \times \mathbb{N} \to A$$

fullfills the postcedent is obvious by the above DIAGRAM.

uniqueness follows as a special instance of Freyd's uniqueness (FR!) : in that scheme take as *initialisation domain* object $\mathbb{1}$, as *iteration domain* object A, as *initialisation map* $a_0 : \mathbb{1} \to A$, and as *endo map* to be *iterated* $f : A \to A$.

Commutativity of the DIAGRAM then shows "existence" and uniqueness of LAWVERE's *sequence* $a : \mathbb{N} \to A$, from the properties of the iterated $f^{\S} : A \times \mathbb{N} \to A$, lower two lines of the DIAGRAM, combined with Freyd's uniqueness (FR!). **q. e. d.**

Comment: Conversely, LAWVERE's NNO is said to have the properties of an NNO in FREYD's version quoted above. But for his proof of this assertion, FREYD relies on internal hom structure with *axiomatic* exponentiation B^A coming with *axiomatic* internal evaluation $\epsilon_{A,B} : B^A \times A \to B$ which is available in his context of an Elementary Topos, not available in present context.[2]

1.4 A monoidal presentation of theory PR

This section is a complement for catgegorically minded readers:

We present the cartesian axioms of fundamental theory **PR** of primitive recursion in terms of *primitive recursive diagonal symmetric* half-*cartesian monoidal structure* ["half" means that the mentioned substitution families, here *terminals* and *projections,* need not to be

[2]in RCF 3 in the References it is shown that the initial cartesian Closed theory with NNO admits code self-evaluation and hence is inconsistent. This is one motivation for not considering here higher order recursion theory. The other motivation is simplicity: the Gödelian case is build on first order in Smorynski 1977.

natural transformations], plus *cartesianness* proper, the latter expressed by

uniqueness of terminal map family $\Pi_A : A \to \mathbb{1}$ and

Godement's equations

$$f = l_{A,B} \circ (f, g) \equiv l_{A,B} \circ (f \times g) \circ \Delta_C,$$
$$g = r_{A,B} \circ (f, g) \equiv r_{A,B} \circ (f \times g) \circ \Delta_C,$$

DIAGRAM

$$\begin{tikzcd}
 & C \arrow[r, "f"] & A \\
C \arrow[ur, "\mathrm{id}"] \arrow[r, "\Delta"'] \arrow[dr, "\mathrm{id}"'] \arrow[rr, bend left, "(f,g)"] & C \times C \arrow[u, "l"'] \arrow[r, "f \times g"'] \arrow[d, "r"] & A \times B \arrow[u, "l"'] \arrow[d, "r"] \\
 & C \arrow[r, "g"'] & B
\end{tikzcd}$$

cf. the diagonal symmetric half-terminal Categories of Budach- & Hoehnke 1975, "realised" in particular as (classical) categories of (sets and) partial maps.

Main reason for this alternative presentation is:

Theories $\mathbf{P\widehat{R}a} \sqsubset \mathbf{P\widehat{R}\mathbb{X}a}$ of (genuine) *partial* p. r. maps to be introduced in chapter 2 inherit the structure of a *PR symmetric diagonal half-cartesian* theory from *basic* p. r. theories $\mathbf{PR\mathbb{X}a} \sqsupset \mathbf{PRa}$ to be discussed below.

[Theory **PRa** is embedding extension of **PR** by *predicate-into-object abstraction.*]

GODEMENT's equations are equivalent to *naturality* of projection family for *Bi*functor $\times : \mathbf{T} \times \mathbf{T} \longrightarrow \mathbf{T}$, $\mathbf{T}$ a cartesian theory.

So here is—alternative—presentation of cartesian part of theory **PR** as a *PR symmetric diagonal half-terminal theory with projections:*

Replace in the cartesian part of presentation of theory **PR** above *formation of the induced* and its *uniqueness* equation (SP) by introduction of the map constants and schemes producing equations below.

Substitution maps:

$\Pi = \Pi_A : A \to \mathbb{1}$, *terminal map* for object A,

$\Theta = \Theta_{A,B} : A \times B \xrightarrow{\cong} B \times A$, *transposition*

$\Delta = \Delta_A : A \to A^2 = A \times A$, *diagonal, duplicate*

$l = l_{A,B} : A \times B \to A$ *left projection,*

$r = r_{A,B} = l_{B,A} \circ \Theta_{A,B} : A \times B \to B \times A \to B$

right projection.

Fundamental for this structure of our theory **PR** is the generation of *cartesian product of maps* by axiom

$$(\times) \quad \frac{f : A \to A',\ g : B \to B' \text{ maps}}{(f \times g) : (A \times B) \to (A' \times B') \text{ map,}}$$

the *cartesian product* of f and g.

As in case of composition, we state an axiom of compatibility of cartesian product of maps with (map) equality, namely

$$(\times_=)\quad \frac{f = f' : A \to A',\ g = g' : B \to B' \text{ maps}}{(f \times g) = (f' \times g') : A \times B \to A' \times B'.}$$

Definability of Δ and Θ by the projections reads

$$(\Theta - \mathrm{proj})\quad \frac{A,\ B \text{ objects}}{\Theta_{A,B} = (r_{A,B}, l_{B,A}) : A \times B \to B \times A,}$$

$$(\Delta - \mathrm{proj})\quad \frac{C \text{ object}}{\begin{array}{l}\Delta_C = (\mathrm{id}, \mathrm{id}) : C \to C \times C, \text{ i.e.} \\ l_{C,C} \circ \Delta = \mathrm{id}_C = r_{C,C} \circ \Delta : \\ C \to C \times C \to C,\end{array}}$$

naturality of *projections* reads:

$$\begin{tikzcd}
A \arrow[r, "f"] & A' \\
A \times B \arrow[u, "l"] \arrow[r, "f \times g"] \arrow[d, "r"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & A' \times B' \arrow[u, "l"'] \arrow[d, "r"'] \\
B \arrow[r, "g"] & B'
\end{tikzcd}$$

cartesian product of maps

We now show the availability of the *induced map* $(f, g) : C \to A \times B$ for given $f : C \to A$ and $g : C \to B$:

Define

$$(f, g) \ =_{\text{def}} \ (f \times g) \circ \Delta_C : C \to C \times C \to A \times B.$$

Then this *induced* obviously fullfills GODEMENT's equations

$$\begin{tikzcd}
 & A \\
C \arrow[ur, bend left, "f"] \arrow[r, "(f{,}g)"] \arrow[dr, bend right, "g"'] & A \times B \arrow[u, "l"'] \arrow[d, "r"'] \\
 & B
\end{tikzcd}$$

uniqueness of the induced map is guaranteed by the earlier equational axiom of surjective pairing

$$
\text{(SP)} \quad \frac{h : C \to A \times B \text{ map}}{\begin{array}{l} h = (l_{A,B} \circ h,\ r_{A,B} \circ h) \\ \ =_{\text{by def}}\ (l_{A,B} \circ h \times r_{A,B} \circ h) \circ \Delta_C : \\ C \xrightarrow{\Delta} C \times C \to A \times B. \end{array}}
$$

These are the axioms and some of the theorems for the *cartesian* Structure of theory **PR**.

A consequence is *compatibility* of *induced map* with *equality:* it follows from compatibility of composition and of cartesian product with equality, combined with the uniqueness of the induced or with distributivity of composition over forming the induced map (LAMBEK).

Cartesian product " $\times$ " introduced above, becomes a *bifunctor*

$$\times : \mathbf{PR} \times \mathbf{PR} \longrightarrow \mathbf{PR}.$$

This follows from the compatibilities with map equation by uniqueness of the induced map: see the following commuting 4 squares rectangular diagram:

$$
\begin{tikzcd}
A \arrow[r, "f"] & A' \arrow[r, "f'"] & A'' \\
A \times B \arrow[u, "l"] \arrow[r, "f \times g"] \arrow[d, "r"] \arrow[rr, bend right, "f' f \times g' g"'] & A' \times B' \arrow[u, "l"] \arrow[r, "f' \times g'"] \arrow[d, "r"] & A'' \times B'' \arrow[u, "l"] \arrow[d, "r"] \\
B \arrow[r, "g"] & B' \arrow[r, "g'"] & B''
\end{tikzcd}
$$

Furthermore it follows the *naturality* of the *substitution* transformations $\Pi_A, \Theta_{A,B}, \Delta_A$.

These are the map term and map-term equality constructions for the cartesian part of theory **PR**, and some of their immediate consequences.

1.5 Introduction of free variables

We start with a ("generic") example of *Elimination* of free variables by their interpretation *into (possibly nested) projections:*

a distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$ gets the map interpretation

$$\begin{aligned}
&a \cdot (b+c) = (a \cdot b) + (a \cdot c) : \\
&R^3 \ =_{\text{by def}} \ R^2 \times R \ =_{\text{by def}} \ (R \times R) \times R \to R, \\
&\qquad \text{with } \textit{systematic} \text{ interpretation of variables:} \\
&a := l\ l\,,\ b := r\ l\,,\ c := r : R^3 = (R \times R) \times R \to R\,,
\end{aligned}$$

and infix writing of operations $op : R \times R \to R$ prefix interpreted as

$$\cdot \circ (a\,, + \circ (b, c)) = + \circ (\cdot \circ (a, b)\,, \cdot \circ (a, c)) : R^3 \to R.$$

In form of a commuting diagram:

$$\begin{tikzcd}
& & (R\times R)\times R \arrow[dll, bend right, "{(a,(b,c))}"', "{(ll,(rl,r))}"] \arrow[ddll, "{(a,b+c)}"] \arrow[ddrr, "{(a\cdot b,a\cdot c)}"'] \arrow[drr, bend left, "{((a,b),(a,c))}", "{((ll,rl),(ll,r))}"'] & & \\
R\times(R\times R) \arrow[d, "R\times +"'] & & & & (R\times R)\times(R\times R) \arrow[d, "\cdot\times\cdot"] \\
R\times R \arrow[drr, "\cdot"] & & & & R\times R \arrow[dll, "+"'] \\
& & R & &
\end{tikzcd}$$

An *iterated* $f^{\S} : A \times \mathbb{N}$ may be written in free-variables notation as

$$f^{\S} = f^{\S}(a, n) = f^{n}(a) : A \times \mathbb{N} \to A$$

with $a := l : A \times \mathbb{N} \to A$, and $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Systematic map interpretation of free-variables Equations:

1. extract the common codomain (domain of values), say B, of both sides of the equation (this codomain may be implicit);

2. "expand" operator priority into additional bracket pairs;

3. transform infix into prefix notation, on both sides of the equation;

4. order the (finitely many) variables appearing in the equation, e.g.lexically;

5. if these variables $a_1, a_2, \ldots, a_{\underline{m}}$ range over the objects $A_1, A_2, \ldots, A_{\underline{m}}$, then fix as common *domain object* (source of commuting diagram), the object

$$A = A_1 \times A_2 \times \ldots \times A_{\underline{m}} \ =_{\text{def}} \ (\ldots((A_1 \times A_2) \times \ldots) \times A_{\underline{m}});$$

6. interpret the variables as *identities* (possibly nested) *projections,* will say: replace, within the equation, all the occurences of a resp. *variable,* by the corresponding—in general *binary nested*—projection;

7. replace each symbol "0" by "$0\ \Pi_D$" where "D" is the (common) domain of (both sides) of the equation;

8. insert composition symbol $\circ$ between terms which are not bound together by an *induced map operator* as in (f_1, f_2);

9. By the above, we have the following two-maps-cartesian-Product rule, forth and back: For

   $a := l_{A,B} : (A \times B) \to A$, $b := r_{A,B} : (A \times B) \to B$, and $f : A \to A'$ as well as $g : B \to B'$, the following identity holds:

$$\begin{aligned}
&(f \times g)(a, b) = (f \times g) \circ (l_{A,B}, r_{A,B}) \\
&= (f \times g) \circ \mathrm{id}_{(A \times B)} = (f \times g) \\
&= (f \circ l_{A,B}, g \circ r_{A,B}) \\
&= (f \circ a, g \circ b) = (f(a), g(b)) : A \times B \to A' \times B';
\end{aligned}$$

10. for free variables $a \in A$, $n \in \mathbb{N}$ interpret the term $f^n(a)$ as the map $f^{\S}(a, n) : A \times \mathbb{N} \to A$.

These 10 interpretation steps transform a (PR) free-variables equation into a variable-free, categorical equation of theory **PR** :

Elimination of (free) variables by their interpretation as *projections,* and vice versa: *Introduction of free variables* as *names* for projections. We allow for mixed notation too, all this, for the time being, only in the context of a cartesian (!) theory **T**.

All of our theories are free from classical, (axiomatic) formal quantification. free variables equations are understood naively as *universally quantified.* But a free variable ($a \in A$) occurring only in the premise of an *implication* takes (in suitable context, see below), the meaning

$$\textit{for any given } a \in A : \text{premise}\,(a, \ldots) \implies \text{conclusion, i.e.}$$
$$\textit{if exists } a \in A \text{ s. t. premise}\,(a, \ldots), \textit{ then } \text{conclusion.}$$

1.6 Goodstein FV arithmetic

In "Development of Mathematical Logic" (Logos Press 1971) R. L. Goodstein gives four basic uniqueness-rules for free-variable Arithmetics. We show here these rules for theory **PR**, and that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction $a \dot{-} n$.

For our evaluation and consistency *considerations below we need from present section equality* predicate $[a \doteq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{2}$, *and that this predicate* ***defines*** *map equality, see equality definability scheme in the middle of section. This scheme is a consequence of commutativity*

$\max(a, b) \ =_{\text{def}} \ a + (b \dot{-} a) = b + (a \dot{-} b) \ =_{\text{by def}} \ \max(b, a)$ *which is difficult to show and which you may take on faith.*

Basic **GA** operations are *addition* '$+$', *predecessor* 'pre', *truncated subtraction* '$\dot{-}$', [in GOODSTEIN *predecessor* written $\mathrm{pre} := (\_) \dot{-} 1$], as well as *multiplication* '$\cdot$'.

We include[3] into Goodstein's uniqueness rules a "passive parameter" a. These extended rules are derivable by use of Freyd's uniqueness theorem (pr!), part of *full scheme* (pr) of primitive recursion which he deduces from his uniqueness (FR!) of the *initialised iterated.*

FREYD 1972 deduces the latter from availability of a natural numbers object $\mathbb{N}$ in LAWVERE'S sense, *axiomatic* availability of higher order *internal* hom objects with, again axiomatic, *evaluation* map family for these objects, of form $\epsilon_{A,B} : B^A \times A \to B$ within the category considered.

Goodstein's rules with passive parameter:

Let $f, g : A \times \mathbb{N} \to \mathbb{N}$ be maps, $s : \mathbb{N} \to \mathbb{N}$ the successor map $n \mapsto n + 1$ and $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ the predecessor map, usually written as $n \mapsto n \dot{-} 1$.

Then Goodstein's rules read:

[3] Sandra Andrasek and the author

$$\mathrm{U}_1 \quad \frac{f(a, sn) = f(a, n) : A \times \mathbb{N} \to B}{f(a, n) = f(a, 0) : A \times \mathbb{N} \to B}$$

no change by application of successor

infers equality *with value at zero* for f

$$\mathrm{U}_2 \quad \frac{f(a, s\,n) = s\,f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) + n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of successors into $+n$

$$\mathrm{U}_3 \quad \frac{f(a, sn) = \mathrm{pre}\ f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) \dot{-} n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of predecessors into $\dot{-}\, n$

$$\mathrm{U}_4 \quad \dfrac{\begin{array}{l} f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = g(a,sn) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n) : A \times \mathbb{N} \to \mathbb{N}}$$

uniqueness of map definition by case-distinction

Rule U_4 is nothing else than *uniqueness* of the *induced map out of the sum* $A \times \mathbb{N} \cong (A \times \mathbb{1}) + (A \times \mathbb{N})$, this sum canonically realised via *injections* $\iota = (\mathrm{id}_A, 0) : A \to A \times \mathbb{N}$ as well as—right injection—$\kappa = \mathrm{id}_A \times s : A \times \mathbb{N} \to A \times \mathbb{N}$.

Proof of these four rules is straight forward for theory **PR**, using FREYD's uniqueness (FR!) and uniqueness clause (pr!) of the *full scheme of primitive recursion* respectively, as follows:

For scheme U_1 consider, with free variable $a := l : A \times \mathbb{N} \to A$,

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A \times s"] \arrow[dd, "f"', "f(a,0)"] & A \times \mathbb{N} \arrow[dd, "f"', "f(a,0)"] \\
A \times \mathbb{N} \arrow[ur, "(a,0)"] \arrow[dr, "f(a,0)"'] & & \\
 & \mathbb{N} \arrow[r, "\mathrm{id}"] & \mathbb{N}
\end{tikzcd}$$

$$(\mathrm{FR!}) \quad \dfrac{}{f(a,n) = f = f(a,0).}$$

Proof of U_2 of *"summing up successors":*

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[d, "f"', "f(a,0)+n"] \arrow[ddr, bend left=70, "f(a,0)+s\,n"] & A \times \mathbb{N} \arrow[d, "f"', "f(a,0)+n"] \\
A \times \mathbb{N} \arrow[ur, "(a,0)"] \arrow[dr, "f(a,0)"'] & & \\
 & \mathbb{N} \arrow[r, "s"] & \mathbb{N}
\end{tikzcd}$$

$$(\mathrm{FR!})\ \frac{\text{pentagon commutative for both } f,\ f(a,0)+n}{f(a,n) = f(a,0)+n}$$

Proof of U_3 is exactly analogous to the above. Replace in statement of U_2 and its proof *stepwise augmentation* $f(a,sn) = s\ f(a,n)$ by *stepwise descent*

$$f(a,s\,n) = f(a,n) \dot{-} 1 \ =_{\text{by def}} \ \text{pre } f(a,n).$$

On right hand side replace *successor* $s : \mathbb{N} \to \mathbb{N}$ by *predecessor* $\text{pre} : \mathbb{N} \to \mathbb{N}$ which in turn is defined by the full scheme (pr) of primitive recursion. In *postcedent* replace *iterated successor* $a+n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by *iterated predecessor* $a \dot{-} n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$.

[In Goodstein's *original,* $\text{pre}(n) = n \dot{-} 1 : \mathbb{N} \to \mathbb{N}$ is a *basic,* "undefined" map constant]

We give a **Direct Proof** of U_4 :

We tailor first this scheme for convenient use of "full" uniqueness scheme (pr!), as follows:

$$U_4 \quad \frac{\begin{array}{l} f = f(a,n),\ f' = f'(a,n) : A \times \mathbb{N} \to B, \\ f(a,0) = f'(a,0) : A \to B, \\ f(a, s\,n) = f'(a, s\,n) : A \times \mathbb{N} \to A \times \mathbb{N} \to B \end{array}}{f = f' : A \times \mathbb{N} \to B.}$$

Choose the *anchor map*

$$g = g(a) := f(a,0) = f'(a,0) :$$
$$A \to A \times \mathbb{N} \to B$$

and the *step map*

$$h = h((a,n),b) := f(a, s\,n) = f'(a, s\,n) :$$
$$(A \times \mathbb{N}) \times B \xrightarrow{l} A \times \mathbb{N} \to B.$$

We obtain, via the *full* scheme (pr!) of PR:

$$(\mathrm{pr!}) \quad \frac{\begin{array}{l} f(a,0) = g(a) = f'(a,0), \ \text{(anchor hypothesis)} \\ f(a, s\,n) = h((a,n), f(a,n)) = f'(a, s\,n) \ \text{(step hypothesis)} \end{array}}{f = \mathrm{pr}[\, g, h\,] = f' : A \times \mathbb{N} \to B \quad \textbf{q. e. d.}}$$

Together with *reflexivity, symmetry,* and *transitivity* of *equality* $f = g : A \to B :$ between maps as well as with the defining *equations* for the fundamental *operations* and $\mathrm{U}_1, \ldots, \mathrm{U}_4$ above, we define categorical Goodstein's **free-variables Arithmetic** which we name **Goodstein Arithmetic, GA**.

1.7 Arithmetical equations

We **quote** here—with *passive parameters* made visible—Goodstein's arithmetical equations together with his **proofs.**

The first equation is (Goodstein's statement numbers)

Lemma:

$$(a \dot{-} n) \dot{-} 1 =^{\mathbf{GA}} (a \dot{-} 1) \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (1.)$$

$a \in \mathbb{N}$ free, "passive", $a := l : A \times \mathbb{N} \to A$,

$n \in \mathbb{N}$ free, *recursive,* $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Proof:

$$\mathrm{U}_3 \quad \frac{(a \dot{-} s\,n) \dot{-} 1 \ =_{\text{by def}}\ ((a \dot{-} n) \dot{-} 1) \dot{-} 1}{(a \dot{-} n) \dot{-} 1 = ((a \dot{-} 0) \dot{-} 1) \dot{-} n}$$

$$=_{\text{by def}}\ (a \dot{-} 1) \dot{-} n : \mathbb{N}^2 \to \mathbb{N} \qquad \textbf{q. e. d.}$$

Next equation is

stepwise simplification rule for truncated subtraction:

$$s\ a \dot{-} s\ b = a \dot{-} b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (1.1)$$

Proof:

$$\mathrm{U}_3 \quad \frac{s\ a \dot{-} s\ s\ b \ =_{\text{by def}}\ (s\ a \dot{-} s\ b) \dot{-} 1}{s\ a \dot{-} s\ b = (s\ a \dot{-} s\ 0) \dot{-} b}$$

$$=_{\text{by def}}\ a \dot{-} b : \mathbb{N}^2 \to \mathbb{N},$$

the latter by definition of the *predecessor* "$\dot{-}\ 1$" **q. e. d.**

Lemma: $a \dot{-} a = 0 : \mathbb{N} \to \mathbb{N}.$ (1.2)

Proof:

$$\mathrm{U}_1 \quad \frac{s\,a \dot{-} s\,a = a \dot{-} a \quad \text{(by stepwise simplification 1.1 above)}}{a \dot{-} a = 0 \dot{-} 0 \ =_{\text{by def}}\ 0}$$ **q. e. d.**

Lemma: $0 \dot{-} a = 0 : \mathbb{N} \to \mathbb{N}.$ (1.3)

Proof:

$$\mathrm{U}_1 \quad \frac{\begin{aligned} 0 \dot{-} s\,a \ &=_{\text{by def}}\ (0 \dot{-} a) \dot{-} 1 \\ &= (0 \dot{-} 1) \dot{-} a \quad \text{(by (1.) above)} \\ &= 0 \dot{-} a : \mathbb{N} \to \mathbb{N} \end{aligned}}{0 \dot{-} a = 0 \dot{-} 0 = 0 : \mathbb{N} \to \mathbb{N}}$$ **q. e. d.**

Proposition:
$a \dot{-} (b + c) = (a \dot{-} b) \dot{-} c : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}.$ (1.31)

Proof:

$$a := l_{\mathbb{N},\mathbb{N}} \circ l_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \xrightarrow{l} \mathbb{N}\times\mathbb{N} \xrightarrow{l} \mathbb{N},$$
$$b := r \circ l : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \xrightarrow{l} \mathbb{N}\times\mathbb{N} \xrightarrow{r} \mathbb{N},$$
$$(a,b) = l_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : A\times\mathbb{N} = \mathbb{N}^2\times\mathbb{N} \xrightarrow{l} A = \mathbb{N}^2,$$
$$c := r : A\times\mathbb{N} = \mathbb{N}^2\times\mathbb{N} \xrightarrow{r} \mathbb{N}.$$

$$(\mathrm{U}_3)\ \frac{\begin{aligned} a \dot{-} (b + s\,c) \ &=_{\text{by def}}\ a \dot{-} s\,(b+c) \quad \text{(definition of +),}\\ &=_{\text{by def}}\ (a \dot{-} (b+c)) \dot{-} 1 \quad \text{(definition of } \dot{-}\text{)}\end{aligned}}{a \dot{-} (b+c) = (a \dot{-} (b+0)) \dot{-} c \ =_{\text{by def}}\ (a \dot{-} b) \dot{-} c.} \quad \textbf{q. e. d.}$$

Full Simplification:

$$(a+n) \dot{-} (b+n) = a \dot{-} b : \mathbb{N}^2\times\mathbb{N} \to \mathbb{N}. \tag{1.4}$$

Proof:

$$(\mathrm{U}_1)\ \frac{\begin{aligned} &(a + s\,n) \dot{-} (b + s\,n)\\ &\quad =_{\text{by def}}\ s\,(a+n) \dot{-} s\,(b+n) = (a+n) \dot{-} (b+n),\\ &\quad \text{by \textit{substitution}---realised essentially as composition}\\ &\quad \text{---of } (a+n) \text{ into } a \text{, and } (a+n) \text{ into } b \text{ within}\\ &\quad \text{\textit{stepwise simplification equation} 1.1 above}\end{aligned}}{(a+n) \dot{-} (b+n) = (a+0) \dot{-} (b+0) \ =_{\text{by def}}\ a \dot{-} b.}$$

Lemma: $0 + n = n \; [\, =_{\text{by def}} \;\; n + 0 \,] \; : \mathbb{N} \to \mathbb{N},$ (2)

Proof:

$$\mathrm{U}_2 \quad \frac{\mathrm{id}_{\mathbb{N}} \, s \, a = s \, a}{\mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a,}$$

and hence

$$a = \mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a = 0 + a : \mathbb{N} \to \mathbb{N} \quad \textbf{q. e. d.}$$

Lemma: $a + s\, b = s\, a + b : \mathbb{N} \times \mathbb{N} \to B.$ (2.1)

Proof by U_2 as follows, with free variable $b := r : \mathbb{N}^2 \to \mathbb{N}$ as *recursion variable:*

For $f = f(a, b) \;\; =_{\text{def}} \;\; a + s\, b : \mathbb{N} \times \mathbb{N} \to N :$

$$U_2 \quad \frac{f(a, s\, b) \;\; =_{\text{by def}} \;\; a + s\, s\, b = s(a + s\, b) = s\, f(a, b) : \mathbb{N}^2 \to \mathbb{N}}{f(a, b) = a + s\, b = f(a, 0) + b}$$

$$=_{\text{by def}} \;\; (a + s\, 0) + b \;\; =_{\text{by def}} \;\; s\, a + b \quad \textbf{q. e. d.}$$

Theorem:

$$a + b = b + a : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (2.2),$$
$$a := l : \mathbb{N}^2 \to \mathbb{N},$$
$$b := r : \mathbb{N}^2 \to \mathbb{N}.$$

Proof:

$$U_4 \quad \frac{\begin{array}{l} a + 0 \ =_{\text{by def}} \ a = 0 + a \text{ by (2) above,} \\ a + s\,b = s\,a + b \text{ by (2.1) above (and symmetry of equality)} \end{array}}{a + b \ =_{\text{by def}} \ f(a, b) = g(a, b)}$$

$$=_{\text{by def}} \ s\,a + b : \mathbb{N}^2 \to \mathbb{N} \quad \textbf{q. e. d.}$$

This gives also sort of
permutability for *truncated subtraction:*

$$(a \dot{-} b) \dot{-} c = (a \dot{-} c) \dot{-} b : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}.$$

Proof:

$(a \dot{-} b) \dot{-} c = a \dot{-} (b + c)$ by (1.31) above
$= a \dot{-} (c + b)$ by commutativity of addition above
$= (a \dot{-} c) \dot{-} b$ again by (1.31) **q. e. d.**

Lemma:

$$(a + n) \dot{-} n = (a + n) \dot{-} (0 + n) = a : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \ \textbf{q. e. d.} \qquad (2.3)$$

Associativity of Addition

$$(a+b)+c = a+(b+c) : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \to \mathbb{N},$$

with free variables

$$a := l \circ l : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} \to \mathbb{N},$$

$$b := r \circ l : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} \to \mathbb{N},$$

$$c := r : (\mathbb{N}\times\mathbb{N})\times\mathbb{N}.$$

Proof: for $f((a,b),c) \ =_{\mathrm{def}} \ a+(b+c) : \mathbb{N}^2\times\mathbb{N} :$

$$U_2 \ \frac{f((a,b),s\,c) = a+(b+s\,c) = a+s(b+c) \\ = s(a+(b+c)) = s\,f((a,b),c)}{a+(b+c) = f((a,b),c) = f((a,b),0)+c \\ =_{\text{by def}} \ (a+(b+0))+c = (a+b)+c : \mathbb{N}^2\times\mathbb{N} \to \mathbb{N}} \quad \textbf{q. e. d.}$$

Recall p. r. **Definition** of *Multiplication:*

$$a \cdot 0 = 0 : \mathbb{N} \to \mathbb{N},$$

$$a \cdot (n+1) = (a \cdot n) + a.$$

For this operation, we have not only *annihilation by zero from the right,* but also

Left zero-Annihilation $0 \cdot n = 0 : \mathbb{N} \to \mathbb{N}.$

Proof:

$$U_1 \ \frac{0 \cdot s\,n = (0 \cdot n) + 0 = 0 \cdot n}{0 \cdot n = 0 \cdot 0 = 0} \quad \textbf{q. e. d.}$$

For proving the other equational laws making the natural numbers object $\mathbb{N}$ into a *unitary commutative semiring* with in addition truncated subtraction introduced above, GOODSTEIN's derived scheme V_4 below is helpfull.

For proof of that scheme, we rely on

Commutativity of maximum operation:[4]

$$\max(a,b) \ =_{\mathrm{def}} \ a+(b \dot{-} a) = b+(a \dot{-} b) \ =_{\mathrm{by\,def}} \ \max(b,a) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$$

Proof: As a first step, we show

Diagonal Reduction Lemma for maximum:

$$\begin{aligned} &\max(a,b) = \max(a \dot{-} 1, b \dot{-} 1) + \mathrm{sign}(a+b) \\ &\ =_{\mathrm{by\,def}} \ \max(a \dot{-} 1, b \dot{-} 1) + (1 \dot{-} (1 \dot{-} (a+b))) : \\ &\mathbb{N}\times\mathbb{N} \to \mathbb{N}, \\ &\max(a, s\,b) = \max(a \dot{-} 1, s\,b \dot{-} 1) + \mathrm{sign}(a + s\,b), \qquad (1) \end{aligned}$$

(where $\mathrm{sign}(0) = 0, \ \mathrm{sign}(s\,n) = 1$), as follows:

$$\max(0, s\,b) = s\,b = \max(0,b) + 1 : \mathbb{N} \to \mathbb{N}, \qquad (2)$$

$$\begin{aligned} &\max(s\,a, s\,b) = s \ \max(a,b) = \max(a,b) + 1 \\ &= \max(s\,a \dot{-} 1, s\,b \dot{-} 1) + \mathrm{sign}(s\,a + s\,b) \qquad (3) \end{aligned}$$

From (2) and (3) follows (1) by uniqueness U_4.

Furthermore

$$\begin{aligned} &\max(a,0) = a = (a \dot{-} 1) + \mathrm{sign}(a) \\ &= \max(a \dot{-} 1, 0 \dot{-} 1) + \mathrm{sign}(a+0). \qquad (4) \end{aligned}$$

[4] in GOODSTEIN 1957 this is taken as an axiom

Together with (1) above, this gives, again by U_4, the **Diagonal Reduction Lemma.**

From this we get immediately by substitution

Opposite Diagonal Reduction Lemma for maximum:

$$\max(b, a) = \max(b \dot{-} 1, a \dot{-} 1) + \operatorname{sign}(b + a)$$
$$= \max(b \dot{-} 1, a \dot{-} 1) + \operatorname{sign}(a + b) \quad \textbf{q. e. d.}$$

Now let

$$\phi = \phi(n, (a, b)) : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N} \text{ by}$$
$$\phi(0, (a, b)) = 0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ and}$$
$$\phi(s\,n, (a, b)) = \phi(n, (a, b)) + \operatorname{sign}((a \dot{-} n) + (b \dot{-} n)) :$$
$$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$$

We show for this *increment* map ϕ

$$\max(a \dot{-} n, b \dot{-} n) + \phi(n, (a, b))$$
$$= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n, (a, b)) \qquad (5)$$

as well as

$$\max(b \dot{-} n, a \dot{-} n) + \phi(n, (a, b))$$
$$= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n, (a, b)) \qquad (6)$$

(same increment).

First we show equation (5): Substitution of $(a \dot{-} n)$ for a and $(b \dot{-} n)$ for b within **Reduction Lemma** above gives

$$\max(a \dot{-} n, b \dot{-} n)$$
$$= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1) + \operatorname{sign}((a \dot{-} n) + (b \dot{-} n))$$

Adding $\phi(n,(a,b))$ to both sides of this equation gives

$$\begin{aligned}
&\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a+b)) \\
&= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1) \\
&\quad + \mathrm{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a+b)) \\
&=_{\text{by def}} \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n,(a,b)),
\end{aligned}$$

i. e. equation (5).

We show equation (6): By substitution of $(b \dot{-} n)$ for b and $(a \dot{-} n)$ for a in **Opposite Reduction Lemma** and addition of $\phi(n,(a,b))$ on both sides, we get

$$\begin{aligned}
&\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)) \\
&= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) \\
&\quad + \mathrm{sign}((b \dot{-} n) + (a \dot{-} n)) + \phi(n,(a,b)) \\
&= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) \\
&\quad + \mathrm{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a,b)) \\
&=_{\text{by def}} \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) + \phi(s\,n,(a,b)) \\
&= \max(b \dot{-} s\,n, a \dot{-} s\,n) + \phi(s\,n,(a,b)),
\end{aligned}$$

i. e. equation (6).

From the two Lemmata, we get by uniqueness U_1

$$\begin{aligned}
&\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b)) \\
&= \max(a \dot{-} 0, b \dot{-} 0) + \phi(0,(a,b)) = \max(a,b) + 0 = \max(a,b) \\
&\quad \text{as well as} \\
&\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)) \\
&= \max(b \dot{-} 0, a \dot{-} 0) + \phi(0,(a,b)) = \max(b,a) + 0 = \max(b,a)
\end{aligned}$$

and hence

$$\max(a,b) = \max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b)) \text{ as well as}$$
$$\max(b,a) = \max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)),$$

and so, by substitution of b into n :

$$\begin{aligned} &\max(a,b) = \max(a \dot{-} b, b \dot{-} b) + \phi(b,a,b) \\ &= (a \dot{-} b) + \phi(b,(a,b)) \\ &= \max(b \dot{-} b, a \dot{-} b) + \phi(b,(a,b)) \\ &= \max(b,a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \end{aligned}$$

q. e. d.

This given, we can now show, for **GA** (and hence for **PR**), scheme

$$\mathrm{V}_4 \quad \frac{\begin{array}{l} f,\ g,\ h : A \times \mathbb{N} \to \mathbb{N} \\ f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = f(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \\ g(a,sn) = g(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n).}$$

Rule V_4 can be derived, by applying rule U_1 to the distance map

$$\begin{aligned} &d(a,n) = |f(a,n), g(a,n)| = |f(a,n) - g(a,n)| \\ &\ =_{\text{by def}}\ (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) : \\ &A \times \mathbb{N} \to \mathbb{N}^2 \xrightarrow{+} \mathbb{N} : \end{aligned}$$

$$\begin{aligned}
d(a,0) &= (f(a,0) \dot{-} g(a,0)) + (g(a,0) \dot{-} f(a,0)) = 0 \\
d(a,sn) &= (f(a,sn) \dot{-} g(a,sn)) + (g(a,sn) \dot{-} f(a,sn)) \\
&= (f(a,n) + h(a,n)) \dot{-} (g(a,n) + h(a,n)) \\
&\quad + (g(a,n) + h(a,n)) \dot{-} (f(a,n) + h(a,n)) \\
&= (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) \\
&= d(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{aligned}$$

whence, by U_1:

$$\begin{aligned}
&d(a,n) = d(a,0) = 0, \text{ i.e.} \\
&(f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) = 0, \text{ whence} \\
&f(a,n) \dot{-} g(a,n) = 0 = g(a,n) \dot{-} f(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{aligned}$$

and hence

$$\begin{aligned}
f(a,n) &= f(a,n) + (g(a,n) \dot{-} f(a,n)) \\
&= \max(f(a,n), g(a,n)) \\
&= \max(g(a,n), f(a,n)) \\
&= g(a,n) + (f(a,n) \dot{-} g(a,n)) \\
&= g(a,n) \quad \textbf{q. e. d.}
\end{aligned}$$

individual equality, equality *predicate*

$$[\, m \doteq n \,] : \mathbb{N}^2 \to \mathbb{2}$$

is defined via weak order as follows:

$$\begin{aligned}
[\, m \leq n\,] \ &=_{\text{def}} \ \neg\,[m \dot{-} n] : \mathbb{N}^2 \to \mathbb{N} \to \mathbb{N}, \ \text{where} \\
\neg\, n \ &=_{\text{def}} \ 1 \dot{-} n, \ \text{directly p. r. defined by} \\
\neg\, 0 \ &=_{\text{def}} \ 1 \equiv \text{true} : \mathbb{1} \to \mathbb{N}, \\
\neg\, s\, n \ &=_{\text{def}} \ 0 \equiv \text{false} : \mathbb{1} \to \mathbb{N}.
\end{aligned}$$

This order on $\mathbb{N}$ is *reflexive* and *transitive.*

Individual equality—first on $\mathbb{N}$—then is easily **defined** by

$$[\, m \doteq n\,] \ =_{\text{def}} \ [\, m \leq n \ \wedge \ n \leq m\,] : \mathbb{N}^2 \to \mathbb{N}.$$

Almost by definition, the triple $\{\leq, \ \doteq, \ \geq\} : \mathbb{N}^2 \to \mathbb{N}$ fullfills the *law of trichotomy,* and $\max(a, b) : \mathbb{N}^2 \to \mathbb{N}$ above is characterised as the *maximum* map with respect to the order $[\, a \leq b\,] : \mathbb{N}^2 \to \mathbb{N}$ just introduced, a posteriori.

We now have at our disposition all ingredients for the

Equality definability theorem:

$$\text{(EqDef)} \quad \frac{\begin{array}{l} f = f(a) : A \to B, \ g = g(a) : A \to B \text{ in } \mathbf{PR}, \\ \mathbf{PR} \vdash \ \text{true}_A = [\, f(a) \doteq_B g(a)\,] : \\ \qquad\qquad A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{\doteq_B} \mathbb{2} \end{array}}{\mathbf{PR} \vdash \ f = g : A \to B, \ \text{i. e. } f =^{\mathbf{PR}} g : A \to B.}$$

Proof:

We begin with the special case $B = \mathbb{N}$: Let $f,\ g : A \to \mathbb{N}$ **PR**-maps satisfying the *antecedent* of (EqDef). Then

$$\begin{aligned}
\mathbf{PR} \vdash f(a) &= f(a) + 0 = f(a) + (g(a) \dot{-} f(a)) \\
&= \max(f(a), g(a)) \\
&= \max(g(a), f(a)) \\
&= g(a) : A \to B.
\end{aligned}$$

The general case for codomain object B follows, since *individual equality* on (binary) cartesian Products is canonically defined *componentwise,* and B is a cartesian product of $\mathbb{N}$'s **q. e. d.**

These *fundamentals* given, we can continue with properties of the algebraic structure on $\mathbb{N}$: all well known classically, they may be taken on faith.

Algebra, Order and Logic on $\mathbb{N}$:

- $\mathbb{N}$ admits the structure

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "s\,0"'] & \mathbb{N} \arrow[d, "\mathrm{sign}", "{>}0"'] \arrow[d, bend right=60, "\neg"'] & \mathbb{N} \times \mathbb{N} \arrow[l, shift left, "\cdot"] \arrow[l, shift right, "+"'] \arrow[l, bend right, "\dot{-}"'] \arrow[dl, shift right, "\leq"'] \arrow[dl, shift left, "\doteq"] \\
& \mathbb{N} &
\end{tikzcd}$$

  of a *unary, commutative semiring with zero*—properties of $\dot{-}$, $\mathrm{sign} : \mathbb{N} \to \mathbb{N}$ ("positiveness"), order, and equality $\doteq$ see below.

- $\mathbb{N}$ admits a foundational important additional algebraic structure, namely *truncated subtraction* $m \dot{-} n : \mathbb{N}^2 \to \mathbb{N}$, with its *simplification properties,* such that multiplication *distributes* over this kind of subtraction.

This distributivity will further entail that of multiplication over "full", not truncated subtraction within

$$\begin{aligned}
&\mathbb{Z} \ =_{\text{def}} \ (\mathbb{N} \times \mathbb{N}) / \doteq_{\mathbb{Z}}, \\
&\qquad \text{with defining equality } \textit{predicate} \\
&[\,(p, q) \doteq_{\mathbb{Z}} (p', q')\,] \ =_{\text{def}} \ [\, p + q' \doteq q + p'\,] : \\
&\mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N} \times \mathbb{N} \xrightarrow{\doteq} \mathbb{N}.
\end{aligned}$$

- $\mathbb{N}$ admits linear *order* $[\, m \leq n\,] : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \subset \mathbb{N}$ as a weak reflexive and transitive *predicate*—this order is p. r. *decidable.*

- As basic logical structures, $\mathbb{N}$ admits *negation*

$$\begin{aligned}
&\neg = \neg\, n : \mathbb{N} \to \mathbb{N}, \ \text{as well as} \\
&\text{sign} = \text{sign}\, n = \neg\,\neg\, n : \mathbb{N} \to \mathbb{N}, \\
&\text{sign}(n) \text{ is directly p. r. defined by} \\
&\text{sign}\, 0 \ =_{\text{def}} \ 0 \equiv \text{false}, \ \text{sign}\, s\, n \ =_{\text{def}} \ 1 \equiv s\, 0 : \\
&\text{sign}\, n = [\, n > 0\,] : \mathbb{N} \to \mathbb{N} \text{ PR decides on } \textit{positiveness.}
\end{aligned}$$

Furthermore, we have a fundamental *equality* predicate

$$\begin{aligned}
&[\, m \doteq n\,] \ =_{\text{by def}} \ [\, m \leq n\,] \ \wedge \ [\, m \geq n\,] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \\
&[\, a \wedge b \ =_{\text{def}} \ \text{sign}(a \cdot b) \text{ logical `and'}\,],
\end{aligned}$$

which is an *equivalence predicate,* and which makes up a *trichotomy* with strict order

$$
\begin{aligned}
&[\,m < n\,] \ =_{\mathrm{def}} \ \mathrm{sign}(n \dot{-} m) \\
&= [\,m \leq n\,] \ \wedge \ \neg\,[\,m \doteq n\,] : \mathbb{N}^2 \to \mathbb{N},
\end{aligned}
$$

Proof of the latter equation is left as an **Exercise.**

- object $\mathbb{N}$ admits definition of (Boolean) "logical functions" by *truth tables,* as does set $\mathbb{2}$ classically—and below in theory **PRa** = **PR** + (abstr) of primitive recursion with predicate abstraction: draw the commuting diagrams.

- **Algebra Combined with Order:** As expected, addition is strongly monotonic in both arguments, multiplication is strongly monotonic for both arguments strictly greater than zero, and truncated subtraction is weakly monotonic in its first argument and weakly antitonic in its second.

Theorem: In free–variables arithmetics the *commutative law* for *multiplication:* $n \cdot m = m \cdot n$, holds.

Proof: We need the following

Lemma:

(i) $0 \cdot n = 0$

(ii) $sa \cdot n = a \cdot n + n$

Proof:

(i) $0 \cdot 0 = 0$ and
$0 \cdot sn = 0 \cdot (n+1) = 0 \cdot n + 0 = 0 \cdot n = 0 \cdot 0 = 0$.

(ii) We show $f(a,n) := sa \cdot n = g(a,n) := a \cdot n + n$ using V_4: $f(a,0) = g(a,0)$ because for $n = 0$ we get $(sa) \cdot 0 = 0$ as well as $a \cdot 0 + 0 = a \cdot 0 = 0$.

$$\begin{aligned}
f(a,sn) &= (sa) \cdot (sn) = (a+1) \cdot (n+1) \\
&= (a+1) \cdot n + (a+1) = (sa) \cdot n + sa \\
&= f(a,n) + h(a,n), \qquad \text{with} \qquad h(a,n) := sa \\
g(a,sn) &= a \cdot (sn) + sn = a \cdot (n+1) + (n+1) \\
&= a \cdot n + a + n + 1 = a \cdot n + n + a + 1 \\
&= a \cdot n + n + sa \\
&= g(a,n) + h(a,n).
\end{aligned}$$

So V_4 gives $f(a,n) = g(a,n)$ i.e. $sa \cdot n = a \cdot n + n$.

q. e. d.

We continue with the proof of $a \cdot n = n \cdot a$:

From $a \cdot 0 = 0 = 0 \cdot a$ and $a \cdot sn = a \cdot n + n = sn \cdot a$ by the Lemma, we conclude $a \cdot n = n \cdot a$ by V_4.[5]

q. e. d.

[5] corrected by S. Lee may 21, 2013

Theorem: In free–variable arithmetics multiplication distributes over addition: $a \cdot (m + n) = a \cdot m + a \cdot n$.

Proof:

Case $n = 0$ is trivial by definition of $+$ and $\cdot$.

From the hypothesis $a \cdot (m+n) = a \cdot m + a \cdot n$ we infer the next step $a \cdot (m + sn) = a \cdot m + a \cdot sn$ by rule V_4 above—with passive parameter (a, m)—as follows:

$$\begin{aligned} \text{with}\quad & f((a,m),n) := a \cdot (m+n), \\ & g((a,m),n) := a \cdot m + a \cdot n \qquad \text{and} \\ & h((a,m),n) := a \end{aligned}$$

we have

$$\begin{aligned} f((a,m),sn) &= a \cdot (m + sn) = a \cdot (m + (n+1)) \\ &= a \cdot ((m+n)+1) = a \cdot (m+n) + a \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= a \cdot m + a \cdot sn = a \cdot m + a \cdot (n+1) \\ &= a \cdot m + a \cdot n + a \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

So by V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m+n) = a \cdot m + a \cdot n$.

q. e. d.

Theorem: In free–variable arithmetics the associative law holds, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

Proof: We prove the law applying rule V_4 with "active" parameter n and passive parameter (a, m) to

$$\begin{aligned} &f((a,m),n) := a \cdot (m \cdot n), \\ &g((a,m),n) := (a \cdot m) \cdot n \quad \text{and} \\ &h((a,m),n) := a \cdot m. \end{aligned}$$

For $n = 0$ we have: $a \cdot (m \cdot n) = a \cdot 0 = 0$, and on the other hand: $(a \cdot m) \cdot 0 = 0$.

For V_4–step we have:

$$\begin{aligned} f((a,m),sn) &= a \cdot (m \cdot sn) = a \cdot (m \cdot (n+1)) \\ &= a \cdot (m \cdot n + m) = a \cdot (m \cdot n) + a \cdot m \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= (a \cdot m) \cdot (n+1) = (a \cdot m) \cdot n + a \cdot m \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

By V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

q. e. d.

Distributivity theorem: In free–variable arithmetics *multiplication distributes* over *truncated subtraction:*

$$a \cdot (m \dot{-} n) = a \cdot m \ \dot{-} \ a \cdot n.$$

Proof by **equality definability,** namely

$$[\, f = g \quad \text{iff} \quad [\, f \doteq g \,] = \mathit{true} \,],$$

it is sufficient to show

$$f((a,m),n) := a \cdot (m \dot{-} n) \doteq a \cdot m \ \dot{-} \ a \cdot n =: g((a,m),n)] = \mathit{true}.$$

Proof of this law becomes comparitively easy with *diagonal induction* out of Pfender, Kröplin, Pape 1994:

Anchoring ($m = 0$ resp. $n = 0$):

$$a \cdot (0 \dot{-} n) = a \cdot 0 = 0 = 0 \dot{-} a \cdot n = a \cdot 0 \ \dot{-} \ a \cdot n, \qquad \text{as well as}$$
$$a \cdot (m \dot{-} 0) = a \cdot m = a \cdot m \dot{-} 0 = a \cdot m \dot{-} a \cdot 0.$$

Diagonal induction **step:**

$$f(a,m,n) := a \cdot (m \dot{-} n) \doteq a \cdot m \dot{-} a \cdot n =: g(a,m,n)$$
$$\implies f(a,sm,sn) = a \cdot (sm \dot{-} sn) \doteq a \cdot sm \dot{-} a \cdot sn = g(a,sm,sn),$$

since

$$\begin{aligned}
f(a,sm,sn) &= a \cdot (sm \dot{-} sn) = a \cdot (m \dot{-} n) \\
&= f(a,m,n), \\
g(a,sm,sn) &= a \cdot sm \dot{-} a \cdot sn = a \cdot (m+1) \dot{-} a \cdot (n+1) \\
&= (a \cdot m + a) \dot{-} (a \cdot n + a) \\
&= a \cdot m \dot{-} a \cdot n \qquad \text{by absorption law for } \dot{-} \\
&= a \cdot (m \dot{-} n) \\
&= g(a,m,n).
\end{aligned}$$

q. e. d.

Proposition: Addition and multiplication in free-variable arithmetics are weakly monotonous, i. e.

$$
\begin{aligned}
m \leq n &\implies m \dot{-} n = 0 \\
&\implies (a+m) \dot{-} (a+n) \doteq 0 \qquad \text{by absorption law for } \dot{-} \\
&\implies a+m \ \leq \ a+n \\
m \leq n &\implies m \dot{-} n = 0 \\
&\implies (a \cdot m) \dot{-} (a \cdot n) \doteq a \cdot (m \dot{-} n) \doteq 0 \\
&\implies a \cdot m \ \leq \ a \cdot n
\end{aligned}
$$

q. e. d.

Boolean Structure on $\mathbb{N}$

In present framework **GA** of Goodstein Arithmetic we introduce on NNO $\mathbb{N}$ the following *proto Boolean* structure:

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "\mathrm{false} \equiv 0"] \arrow[r, shift right, "\mathrm{true} \equiv 1"'] & \mathbb{N} \arrow[d, bend right, "\neg"'] \arrow[d, ">0 \mid \mathrm{sign}"] & \mathbb{N} \times \mathbb{N} \arrow[l, shift right, "\vee \equiv +"'] \arrow[l, shift left, "\wedge \equiv \cdot"] \arrow[dl, bend left, "\Rightarrow \equiv \leq"'] \arrow[dl, bend left=40, "\Leftrightarrow \equiv \doteq"] \\
& \mathbb{N} &
\end{tikzcd}
$$

[Successors are all viewed logically to represent truth value true.]

1.8 Sum objects and definition by distinction of cases

We want to construct here in variable-free manner map definition

$$f = f(a) = \mathrm{if}[\chi, (g|h)](a) = \begin{cases} g(a) \text{ if } \chi(a) \\ h(a) \text{ if } \neg\,\chi(a) \textit{ (otherwise).} \end{cases} :$$

$$A \to B$$

by *case distinction* for given $f, g : A \to B$ and predicate $\chi : A \to \mathbb{2}$.

We use a special *sum* diagram,

"Hilbert's infinite hotel" $\mathbb{N} \cong \mathbb{1} + \mathbb{N}$:

We consider the **sum** diagram

$$\begin{tikzcd}
A \times \mathbb{1} \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "a\times 0"'] & A \arrow[d, "{(a,0)}"'] \arrow[dr, bend left, "f"] & \\
& A \times \mathbb{N} \arrow[r, dotted, "(f|g)"] & B \\
& A \times \mathbb{N} \arrow[u, "a\times s"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

where

$$(f|g) \;=_{\mathrm{def}}\; \mathrm{pr}[f : A \to B,\;\, g \circ l : (A \times \mathbb{N}) \times B \to A \times \mathbb{N} \to B]$$

is the *unique* commutative fill-in into this *sum diagram:* full scheme (pr) of primitive recursion. Symbolically:

$$A \times \mathbb{N} = A + (A \times \mathbb{N}) \cong (A \times \mathbb{1}) + (A \times \mathbb{N}).$$

An important consequence then is in fact the following scheme of map definition by **case distinction:**

$$
\text{(IF)} \quad \frac{\begin{array}{l} \chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \text{ p. r. predicate,} \\ g, h : A \to B \text{ p. r. maps} \end{array}}{\begin{array}{l} f = \mathrm{if}[\chi, (g|h)] \text{ “if } \chi \text{ then } g \text{ else } h\text{”} \\ \;=_{\mathrm{def}} \; (h|g \circ l) \circ (\mathrm{id}_A, \chi) : \\ A \to A \times \mathbb{N} \to B, \\ \chi(a) \implies \mathrm{if}[\chi, (g|h)] \doteq g(a), \\ \neg\chi(a) \implies \mathrm{if}[\chi, (g|h)] \doteq h(a). \end{array}}
$$

Proof: Commuting DIAGRAM:

$$
\begin{tikzcd}
& A \arrow[d, "(\mathrm{id},0)"] \arrow[rrd, bend left, "h"] & & \\
A \arrow[r, "(\mathrm{id},\chi)"] \arrow[rrr, bend right, "f"'] & A \times \mathbb{N} \arrow[rr, dotted, "(h|g\circ l)"] & & B \\
& A \times \mathbb{N} \arrow[u, "A\times s"] \arrow[r, "l"'] & A \arrow[ru, bend right, "g"'] &
\end{tikzcd}
$$

with $(h|g\,l) : A \times \mathbb{N} = A + (A \times \mathbb{N}) \to B$ the induced map out of the sum (“coproduct”), coproduct *injections* $(\mathrm{id}, 0), A \times s$. The two implications for $f = \mathrm{if}[\chi, (g|h)]$ follow **q. e. d.**

1.9 Substitutivity and Peano induction

Leibniz substitutivity theorem for predicative equality reads:

$$\frac{f : A \to B \ \ \mathbf{PR}\text{-map}}{\begin{array}{l} a \doteq a' \implies f(a) \doteq f(a') : \\ A \times A \to \mathbb{N}. \end{array}}$$

Proof by structural induction on f :

- $f = 0 : \mathbb{1} \to \mathbb{N}$: clear since $0 \doteq 0 : \mathbb{1} \to \mathbb{N} \times \mathbb{N} \xrightarrow{\doteq} \mathbb{N}$.
- $f = s : \mathbb{N} \to \mathbb{N}$: Use $[s\,m \dot{-} s\,n] = [m \dot{-} n]$ and

  $[a \doteq b] = [a \leq b] \wedge [b \leq a] = \neg[a \dot{-} b] \wedge \neg[b \dot{-} a]$.
- $f = \Pi : A \to \mathbb{1}$: trivial since $\doteq_{\mathbb{1}} = \ \mathrm{true}_{\mathbb{1} \times \mathbb{1}}$.
- $f = l : A \times B \to A$:

  $(a, b) \doteq (a', b') \iff [a \doteq a'] \wedge [b \doteq b']$

  $\implies [a \doteq a'] \iff [l(a, b) \doteq l(a', b')]$:

  $(A \times B) \times (A \times B) \to \mathbb{N}$.
- $f = r : A \times B \to B$: analogous.

  Further recursively:

- for a composition $g \circ f : A \to B \to C$:

$$\begin{aligned}
&a \doteq a' \implies f\,a \doteq f\,a' \ (\textit{hypothesis})\\
&\implies g(f\,a) \doteq g(f\,a') \ (\textit{hypothesis})\\
&\iff (g \circ f)(a) \doteq (g \circ f)(a') : A \times A \to \mathbb{N}.
\end{aligned}$$

- for an induced $(f, g) : C \to A \times B$:

$$\begin{aligned}
&c \doteq c' \implies f(c) \doteq f(c') \wedge g(c) \doteq g(c') \ (\textit{hypothesis})\\
&\iff (f(c), g(c)) \doteq (f(c'), g(c'))\\
&\iff (f, g)(c) \doteq (f, g)(c') : C \times C \to \mathbb{N}.
\end{aligned}$$

- for an iterated map $f^{\S} : A \times \mathbb{N} \to A$ to show:

$$(a, n) \doteq (a', n') \implies f^{\S}(a, n) \doteq f^{\S}(a', n) : (A \times \mathbb{N})^2 \to A.$$

Diagonal induction on $(n, n') \in \mathbb{N} \times \mathbb{N}$:

$(a, 0) \doteq (a', 0) \implies f^{\S}(a, 0) \doteq a \doteq a' \doteq f^{\S}(a', 0)$;

left axis: $(a, 0) \neq (a, s\,\mathrm{pre}(n'))$, premise fails;

right axis: $(0, a') \neq (s\,\mathrm{pre}(n), a')$, premise fails;

diagonal induction step:

$$\begin{aligned}
&(a, s\,n) \doteq (a', s\,n') \implies a \doteq a' \wedge s\,n \doteq s\,n'\\
&\implies a \doteq a' \wedge n \doteq n' \ (\textit{injectivity of } s)\\
&\implies (a, n) \doteq (a', n') \implies f^{\S}(a, n) \doteq f^{\S}(a', n')\\
&\qquad (\textit{induction hypothesis})\\
&\implies f^{\S}(a, s\,n) \doteq f(f^{\S}(a, n)) \doteq f(f^{\S}(a', n')) \doteq f^{\S}(a', s\,n')
\end{aligned}$$

(*structural recursion hypothesis on* f)

q. e. d.

Peano's axioms read in categorical free-variables form:[6]

Peano theorem:

- P1: *zero is a natural number:*

  $0 : \mathbb{1} \to \mathbb{N}$ is a map constant of $\mathbb{N}$, a *natural number* as such.

  [Other natural numbers are free variables on $\mathbb{N}$]

- P2: *to any natural number (free variable) n is assigned* a *successor:*

  This *assignment* is realised categorically by *successor map*

  $s = s(n) : \mathbb{N} \to \mathbb{N}$.

  Such successor $s(n)$ is *unique:*

  This is given categorically by LEIBNIZ's substitutivity for the successor map:

  $$\mathbf{PR} \vdash \; m \doteq n \implies s(m) \doteq s(n) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

- P3: 0 *is not a successor:*

  This follows from $sn > 0$, whence $sn \neq 0$, by definition of $m \doteq n$ via $m < n$ via $m \dot{-} n$.

- P4: *equality* $s(m) \doteq s(n)$ *implies* $m \doteq n$:

[6] see REITER 1982 as well as PFENDER, KRÖPLIN & PAPE

This is derived *injectivity* of successor map $s : \mathbb{N} \to \mathbb{N}$ which reads in free variables:

$$\begin{aligned} & s\,m \equiv s(m) \doteq s(n) \equiv s\,n \\ & \quad \implies m \doteq \mathrm{pre}\,s\,m \doteq \mathrm{pre}\,s\,n \doteq n : \\ & \mathbb{N} \times \mathbb{N} \to \mathbb{N}. \end{aligned}$$

- P5: Peano-***induction***, derived from *uniqueness* part (pr!) of *full* scheme (pr) of primitive recursion (Freyd):

$$(\mathrm{P5}) \quad \frac{\begin{array}{l} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{N} \quad \text{predicate} \\ \varphi(a,0) = \mathrm{true}_A(a) \quad (\mathit{anchor}) \\ [\,\varphi(a,n) \implies \varphi(a,s\,n)\,] = \mathrm{true}_{A\times\mathbb{N}} \quad (\mathit{induction\ step}) \end{array}}{\varphi(a,n) = \mathrm{true}_{A\times\mathbb{N}} \qquad (\mathit{conclusio}).}$$

Proof of Peano induction principle (P5) from *full scheme* (pr) of primitive recursion:[7]

For scheme (pr!) choose as anchor map

$$\begin{aligned} & g = g(a) = \varphi(a,0) = \mathrm{true}(a) : A \to \mathbb{N}, \text{and as step map} \\ & h = h((a,n),b) = b \vee \varphi(a,s\,n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \end{aligned}$$

By (pr) we get a unique $f = f(a,n) : A \times \mathbb{N} \to \mathbb{N}$ which satifies

$$\begin{aligned} & f(a,0) = \varphi(a,0) = \mathrm{true}(a) \quad \text{and} \\ & f(a,s\,n) = h((a,n),f(a,n)) = f(a,n) \vee \varphi(a,s\,n). \end{aligned}$$

[7] Reiter 1982 and Pfender, Kröplin, Pape 1994

This works for $f = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$ as well as for $f = \varphi$, the latter since

$$\begin{aligned}
&\varphi(a,n) \vee \varphi(a,s\,n) \\
&= (\varphi(a,n) \vee \varphi(a,s\,n)) \wedge (\varphi(a,n) \Rightarrow \varphi(a,s\,n)) \\
&\quad \text{by 2nd hypothesis} \\
&= \varphi(a,s\,n) \quad \text{by boolean tautology} \\
&(\alpha \vee \beta) \wedge (\alpha \Rightarrow \beta) = \beta : \\
&\textit{test with } \beta = 0 \equiv \mathrm{false} \textit{ and } \beta = 1 \equiv \mathrm{true}.
\end{aligned}$$

q. e. d.

By replacing predicate φ with

$$\psi(a,n) := \bigwedge_{i \le n} \varphi(a,i) : A \times \mathbb{N} \to \mathbb{N}$$

in this proof we get

Course of values induction principle:

$$\text{(P5)} \quad \frac{\begin{array}{ll} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{N} & \text{predicate} \\ \varphi(a,0) = \mathrm{true}_A(a) & (\textit{anchor}) \\ \big[\bigwedge_{i \le n} \varphi(a,i) \implies \varphi(a,s\,n)\big] = \mathrm{true}_{A\times\mathbb{N}} & (\textit{induction step}) \end{array}}{\varphi(a,n) = \mathrm{true}_{A\times\mathbb{N}} \quad (\textit{conclusio}).}$$

Here predicate $\bigwedge_{i \leq n} \varphi(a,i) : A \times \mathbb{N} \to \mathbb{N}$ is p. r. **defined** by

$$\begin{aligned}
&\bigwedge_{i \leq 0} \varphi(a,i) = \varphi(a,0) : A \to \mathbb{N}, \\
&\bigwedge_{i \leq s\,n} \varphi(a,i) = \bigwedge_{i \leq n} \varphi(a,i) \wedge \varphi(a, s\,n) : A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}$$

1.10 Integer division and related

Integer division with remainder (Euclide)

$$(a \div b, a \text{ rem } b) : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N} \times \mathbb{N}$$

is characterised by

$$\begin{aligned}
&a \div b = \max\{c \leq a : b \cdot c \leq a\} : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}, \\
&a \text{ rem } b = a \dot{-} (a \div b) \cdot b : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}.
\end{aligned}$$

[for $\mathbb{N}_{>} = \{n \in \mathbb{N} : n > 0\}$ and objects defined by p. r. predicate abstraction in general see next chapter.]

Explicitely, we **define**

$$\div = a \div b : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}$$

via *initialised iteration* $h = h((a,b),n)$ of

$$g = g((a,b),c) = \begin{cases} ((a,b),c) \text{ if } a < b, \\ ((a \dot{-} b, b), c+1) \text{ if } a \geq b \end{cases}$$

in

$$\begin{tikzcd}
 & (\mathbb{N}\times\mathbb{N}_>)\times\mathbb{N} \arrow[r, "(N\times\mathbb{N}_>)\times s"] \arrow[dd, dotted, "h"] & (\mathbb{N}\times\mathbb{N}_>)\times\mathbb{N} \arrow[dd, dotted, "h"] \\
\mathbb{N}\times\mathbb{N}_> \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "(\mathrm{id},0)"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
 & (\mathbb{N}\times\mathbb{N}_>)\times\mathbb{N} \arrow[r, "g"] & (\mathbb{N}\times\mathbb{N}_>)\times\mathbb{N}
\end{tikzcd}$$

$$\begin{aligned}
a \div b &=_{\mathrm{def}} \ r\,h((a,b),a) : \mathbb{N}\times\mathbb{N}_> \to (\mathbb{N}\times\mathbb{N}_>)\mathbb{N} \to \mathbb{N}, \\
a \,\mathrm{rem}\, b &=_{\mathrm{def}} \ ll\,h((a,b),a) = a \dot{-} b \cdot (a \div b) : \mathbb{N}\times\mathbb{N}_> \to \mathbb{N}.
\end{aligned}$$

The predicate $a|b : \mathbb{N}_> \times \mathbb{N} \to \mathbb{N}$, *$a$ is a divisor of b*, *a divides b* is **defined** by

$$a|b = [\, b \text{ rem } a \doteq 0\,].$$

Exercise: Construct the Gaussian algorithm for determination of the *gcd* of $a, b \in \mathbb{N}_>$ **defined** as

$$\mathrm{gcd}(a,b) = \max\{c \le \min(a,b) : c|a \wedge c|b\} : \mathbb{N}_> \times \mathbb{N}_> \to \mathbb{N}_>$$

by iteration of mutual rem.

Primes

Define the predicate *is a prime* by

$$\mathbb{P}(p) = \bigwedge_{m=1}^{p} [m|p \Rightarrow m \doteq 1 \vee m \doteq p] : \mathbb{N} \to \mathbb{2} :$$

Only 1 and p divide p.

Write $\mathbb{P}$ for $\{n \in \mathbb{N} : \mathbb{P}(n)\} \subset \mathbb{N}$ too.

The (euclidean) count $p_n : \mathbb{N} \rightarrowtail \mathbb{N}$ of all primes is given by

$$\begin{aligned} p_0 &= 2, \\ p_{n+1} &= \min\{p \in \mathbb{N} : \mathbb{P}(p), p_n < p \leq \prod_q [q \leq p_n \wedge \mathbb{P}(q)]\} + 1 \\ &= \min\{p \in \mathbb{N} : \mathbb{P}(p), p < 2p_n\} : \\ &\mathbb{P} \rightarrowtail \mathbb{P}, \end{aligned}$$

iterated binary product and iterated binary minimum.

The latter presentation is given by BERTRAND's theorem.

Notes

(a) An NNO, within a cartesian Closed category of sets, was first studied by Lawvere 1964.

(b) Eilenberg-Elgot 1970 iteration, here special case of one-successor iteration theory **PR**, is because of Freyd's uniqueness scheme (FR!) a priori stronger than classical free-variables *primitive recursive arithmetic* **PRA** in the sense of SMORYNSKI 1977. If viewed as a conservative subsystem of **PM**, **ZF** or **NGB**, that **PRA** is stronger than our **PR**.

(c) Within Topoi (with their cartesian closed structure), Freyd 1970 characterised Lawvere's NNO by unique initialised iteration. Such Freyd's NNO has been called later, e.g. in Maietti 2010, *parametrised NNO.*

(d) Lambek-Scott 1986 consider in parallel a *weak NNO:* uniqueness of Lawvere's sequnces $a : \mathbb{N} \to A$ not required. We need here uniqueness (of the initialised iterated) for proof of Goodstein's 1971 uniqueness rules basic for his development of p. r. arithmetic. Without the latter uniqueness requirement, the definition of parametrised (weak) NNO is equational.

(e) For uniqueness of the set of natural numbers (out of the Peano-axioms), classical set theory needs *higher order.* This corresponds in category theory to the use of free meta-variables on *maps.*

In first order classical, elementhood based Peano-arithmetic there are other models of the natural numbers, even uncountable ones. Others than the "standard" (e.g. von Neumann) model.

Chapter 2

Predicate Abstraction

We extend the fundamental theory **PR** of primitive recursion *definitionally* by predicate abstraction objects $\{A : \chi\} = \{a \in A : \chi(a)\}$. We get an (embedding) extension $\mathbf{PR} \sqsubset \mathbf{PRa}$ having all of the expected properties. You may just digest the **Structure theorem** for theory **PRa** below, theory of *primitive recursion with scheme of predicate abstraction.*

2.1 Extension by predicate abstraction

We discuss a p. r. **abstraction scheme** as a definitional enrichment of **PR**, into theory **PRa** of *PR decidable objects and PR maps in between,* decidable subobjects of the objects of **PR**. The objects of **PR** are, up to isomorphism,

$$\mathbb{1},\ \mathbb{N}^1 =_{\text{def}} \mathbb{N},\ \mathbb{N}^{\underline{m}+1} =_{\text{def}} (\mathbb{N}^{\underline{m}} \times \mathbb{N}).$$

[$\underline{m}$ is a free metavariable, over the NNO constants $0, 1 = s\,0,\ 2 = s\,s\,0, \ldots \in \underline{\mathbb{N}}$.]

The extension **PRa** is given by adding schemes ($\mathrm{Ext}_{\mathbf{Obj}}$), ($\mathrm{Ext}_{\mathbf{Map}}$), and ($\mathrm{Ext}_{=}$) below. Together they correspond to the *scheme of abstraction* in **set** theory, and they are referred below as *schemes* of *PR abstraction.*

Our first predicate-into-object *abstraction* scheme is

$\chi : A \to \mathbb{N}$ a **PR**-predicate:

$\mathrm{sign} \circ \chi = \chi : A \to \mathbb{N} \to \mathbb{N}$,

$$\begin{tikzcd} Au \arrow[r, "\chi"] \arrow[rr, bend right=50, "\chi"', "="] & \mathbb{N} \arrow[r, "\mathrm{sign}"] & \mathbb{N} \end{tikzcd}$$

($\mathrm{Ext}_{\mathbf{Obj}}$) ———————————————

$\{A : \chi\}$ object (of emerging theory **PRa**)

Subobject $\{A : \chi\} \subseteq A \cong \mathbb{N}^{\underline{n}}$ may be written alternatively, with *bound* variable a, as

$$\{A : \chi\} = \{a \in A : \chi(a)\}.$$

$\{A : \chi\}$ is just another name for the (external) code $\chi \in \mathbf{PR} \subset \underline{\mathbb{N}}$, a NNO constant out of $\underline{\mathbb{N}}$, the external set of natural number constants

$$0,\ 1 \equiv s\,0,\ 2 \equiv s\,s\,0 \text{ etc. } \underline{n} \equiv s\,\ldots\,s\,0 \equiv \mathrm{num}(\underline{n}) \in \underline{\mathbb{N}} \text{ etc.}$$

The *maps* of **PRa** = **PR** + (abstr) come in by

$$(\mathrm{Ext}_{\mathbf{Map}})\quad \frac{\begin{array}{l}\{A:\chi\},\ \{B:\varphi\}\ \mathbf{PRa}\text{-objects},\\ f:A\to B \text{ a } \mathbf{PR}\text{-map},\\ \mathbf{PR}\vdash\ \chi(a)\implies\varphi\, f\,(a), \text{ i.e.}\\ [\,\chi\implies\varphi\circ f\,] =^{\mathbf{PR}} \mathrm{true}_A : A\xrightarrow{\Pi}\mathbb{1}\xrightarrow{1}\mathbb{N}\end{array}}{f \text{ is a } \mathbf{PRa}\text{-map } f:\{A:\chi\}\to\{B:\varphi\}}$$

In particular, if for predicates $\chi', \chi'' : A \to \mathbb{N}$

$\mathbf{PR} \vdash\ \chi'(a) \implies \chi''(a) : A \to \mathbb{N}\times\mathbb{N} \to \mathbb{N}$,
then $\mathrm{id}_A : \{A : \chi'\} \to \{A : \chi''\}$ in **PRa** is called an *inclusion,*
and written $\quad \subseteq : A' = \{A : \chi'\} \to A'' = \{A : \chi''\}$ or $A' \subseteq A''$.

Nota bene: For predicate (terms!) $\chi, \varphi : A \to \mathbb{N}$ such that $\mathbf{PR} \vdash \chi = \varphi : A \to \mathbb{N}$ (logically: such that $\mathbf{PR} \vdash\ [\chi \iff \varphi])$ we have

$$\{A:\chi\} \subseteq \{A:\varphi\} \text{ and } \{A:\varphi\}\subseteq\{A:\chi\},$$

but—in general—not *equality of objects.* We only get in this case

$$\mathrm{id}_A : \{A:\chi\} \xrightarrow{\cong} \{A:\varphi\}$$

as an **PRa** *isomorphism.*

A posteriori, we introduce as Reiter does, the formal *truth Algebra* $\mathbb{2}$ as

$$\mathbb{2}\ =_{\mathrm{def}}\ \{n\in\mathbb{N}:\chi(n)\},\ \text{where } \chi(n) = [n\le 1]:\mathbb{N}\to\mathbb{N},$$

with proto Boolean operations on $\mathbb{N}$ restricting—in codomain and domain—to *boolean* operations on $\mathbb{2}$ resp.

$$\mathbb{2} \times \mathbb{2} \ =_{\text{def}} \ \{(m, n) \in \mathbb{N} \times \mathbb{N} : m, n \leq s\,0\},$$

by definition below of cartesian Product of objects within **PRa**.

PRa-maps with common **PRa** domain and codomain are considered equal, if their values are equal on their defining *domain predicate.* This is expressed by the scheme

$$(\mathrm{Ext}_=) \quad \frac{\begin{array}{l} f,\, g : \{A : \chi\} \to \{B : \varphi\} \ \textbf{PRa}\text{-maps}, \\ \mathbf{PR} \vdash \ \chi(a) \implies f(a) \doteq_B g(a) \end{array}}{f = g : \{A : \chi\} \to \{B : \varphi\},}$$

explicitly:

$$\begin{array}{l} f =^{\mathbf{PRa}} g : \{A : \chi\} \to \{B : \varphi\}, \text{ also noted} \\ \mathbf{PRa} \vdash \ f = g : \{A : \chi\} \to \{B : \varphi\}. \end{array}$$

Structure Theorem for the theory **PRa** of *primitive recursion with predicate abstraction:* [1]

PRa is a cartesian p. r. theory. The theory **PR** is cartesian p. r. embedded. The theory **PRa** has universal extensions of all of its predicates and a boolean truth object as codomain of these predicates, as well as map definition by case distinction. In detail:

[1] cf. Reiter 1980

(i) **PRa** inherits associative map composition and identities from **PR**.

(ii) **PRa** has **PR** fully embedded by

$$\langle f : A \to B \rangle \mapsto \langle f : \{A : \mathrm{true}_A\} \to \{B : \mathrm{true}_B\} \rangle$$

(iii) **PRa** has cartesian product

$$\{A : \chi\} \times \{B : \varphi\} \ =_{\mathrm{def}} \ \{A \times B : \chi \wedge \varphi : A \times B \to \mathbb{N} \times \mathbb{N} \xrightarrow{\wedge} \mathbb{N}\},$$

with projections and universal property inherited from **PR**.

We abbreviate $\{A : \mathrm{true}_A\}$ by A.

(iv) object $\mathbb{2}$ comes as a *sum* $\mathbb{1} \xrightarrow[0]{\mathrm{false}} \mathbb{2} \cong \mathbb{1} + \mathbb{1} \xleftarrow[1]{\mathrm{true}} \mathbb{1}$ over which cartesian product $A \times \_$ *distributes.*

This allows in fact for the usual truth-table definitions of all boolean operations on object $\mathbb{2}$ and for p. r. map definition by case distinction.

(v) The embedding $\sqsubset : \mathbf{PR} \longrightarrow \mathbf{PRa}$ is a *cartesian functor :* it preserves products and their cartesian universal property with respect to the projections inherited from **PR**.

(vi) **PRa** has *extensions* of its predicates, namely

$$\mathrm{Ext}\,[\,\varphi : \{A : \chi\} \to \mathbb{2}\,] \ =_{\mathrm{def}} \ \{A : \chi \wedge \varphi\} \subseteq \{A : \chi\},$$

characterised as (**PRa**)-*equalisers*

$$\mathrm{Equ}\,(\chi \ \wedge \ \varphi, \ \mathrm{true}_A) : \{A : \chi\} \to \mathbb{2}$$

[mutatis mutandis: within theory **PRa**, we identify predicates $\chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \to \mathbb{N}$ with maps $\chi : A \to \mathbb{2}$.]

(vii) **PRa** has all *equalisers,* namely equalisers

$$\begin{aligned} &\mathrm{Equ}[\,f,g\,] \ =_{\mathrm{def}} \ \{a \in A : \chi(a) \,\wedge\, f(a) \doteq_B g(a)\} \\ &= \mathrm{Ext}[\,\doteq_B \circ\, (f,g) : A' \to B' \times B' \xrightarrow{\doteq} \mathbb{2}\,], \end{aligned}$$

of arbitrary **PRa** map pairs $f, g : A' = \{A : \chi\} \to B' = \{B : \varphi\}$, and hence all finite projective *limits,* in particular *pullbacks* which we will rely on later.

A *pullback,* of a map $f : A \to C$ *along* a map $g : B \to C$, also of g along f, is the square in

$$\begin{tikzcd}
D \arrow[dr, dotted, "{(h,k)}"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& P \arrow[r, "g'"'] \arrow[d, "f'"] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

[I prefer this "set theoretical" way to construct extension sets out of the cartesian category structure of fundamental theory **PR**, and then I construct equalisers and the other finite limits on this basis. Another possibility—Romàn1989—is to add equalisers as *undefined notion* and to construct directly from these and cartesian product. The relation between (vi) and (vii) is best understood set theoretically: use free variable argument chase, and recall set theoretical definition of an equaliser.]

The embedding preserves such limits as far as available already in **PR**. Equality *predicate* extends to cartesian Products componentwise as

$$[\,(a,b) \doteq_{A\times B} (a',b')\,] \;=_{\mathrm{def}}\; [\,a \doteq_A a'\,] \wedge [\,b \doteq_B b'\,] : (A\times B)^2 \to \mathbb{2},$$

and to (predicative) subobjects $\{A : \chi\}$ by restriction.

(viii) arithmetical structure extends from **PR** to **PRa**, i.e. **PRa** admits the *iteration* scheme as well as FREYD's *uniqueness* scheme: the iterated

$$f^{\S} : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\} \to \{A : \chi\}$$

is just the *restricted* **PR**-map $f^{\S} : A \times \mathbb{N} \to A$, the uniqueness schemes follow from definition of $=^{\mathbf{PRa}}$ via **PRa**'s scheme $(\mathrm{Ext}_=)$ above.

(ix) In particular, our *equality predicate* $\doteq_A : A^2 \to \mathbb{N}$, restricted to subobjects $A' = \{A : \chi\} \subseteq A$, inherits all of the properties of equality on $\mathbb{N}$ and the other *fundamental objects.*

(x) **Countability:** Each fundamental object A i.e. A a finite power of $\mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$, admits, by CANTOR's isomorphism

$$\mathrm{ct} = \mathrm{ct}_{\mathbb{N}\times\mathbb{N}}(n) : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N},$$

a retractive count $\mathrm{ct}_A(n) : \mathbb{N} \to A$.

Problem: For which predicates $\chi : A \to \mathbb{2}$ (A fundamental) does theory **PRa** admit a retractive *count*

$$\mathrm{ct} = \mathrm{ct}_{\{A:\chi\}}(n) : \mathbb{N} \to \{A : \chi\}?$$

The difficulty is seen already in case $\emptyset_A \;=_{\text{by def}}\; \{A : \mathrm{false}_A\}$. A *sufficient condition* is $\{A : \chi\}$ to come with a *point,* $a_0 : \mathbb{1} \to \{A : \chi\}$. But there may be non-empty objects without points in some "ill", ω*-inconsistent* theories.

Proof of PRa Structure Theorem

This proof is *general abstract nonsense,* the **theorem** can be taken on faith. Here is that **proof:**

(i) For $f : \{A : \chi\} \to \{B : \varphi\},\ g : \{B : \varphi\} \to \{C : \psi\}$ in **PRa** we have

$$\mathbf{PR} \vdash \chi \implies \varphi\, f \implies \psi\, g\, f : A \to \mathbb{N},$$

whence $g \circ f : \{A : \chi\} \to \{C : \psi\}$ in **PRa**, asociativity of composition and neutrality of identitities are inherited from **PR**.

Compatibility of composition with $=^{\mathbf{PRa}}$: For

$$f =^{\mathbf{PRa}} f' : \{A : \chi\} \to \{B : \varphi\},$$
$$g =^{\mathbf{PRa}} g' : \{B : \varphi\} \to \{C : \psi\} \text{ in } \mathbf{PRa}$$

we show

$$g \circ f =^{\mathbf{PRa}} g \circ f' : \{A : \chi\} \to \{C : \psi\},$$
$$g' \circ f =^{\mathbf{PRa}} g \circ f : \{A : \chi\} \to \{C : \psi\} :$$

$$\frac{\mathbf{PR} \vdash \chi(a) \implies f(a) \doteq_B f'(a) : A \to \mathbb{N}}{\mathbf{PR} \vdash \chi(a) \implies g\, f(a) \doteq_C g\, f'(a) : A \to \mathbb{N},}$$

$$\begin{array}{l}
\mathbf{PR} \vdash \chi(a) \implies \varphi\, f(a) : A \to \mathbb{N} \\
\mathbf{PR} \vdash \varphi(b) \implies g(b) \doteq_C g'(b) : A \to \mathbb{N} \\
\hline
\mathbf{PR} \vdash \chi(a) \implies g\, f(a) \doteq_C g'\, f(a) : A \to \mathbb{N}
\end{array}$$

both by Leibniz substitutivity with respect to $\doteq$ q. e. d.

The embedding assertion (ii) is obvious.

Assertion (iii), cartesian product: Consider induced-into-product DIAGRAM

$$\begin{tikzcd}
 & \{A : \chi\} \\
\{C : \psi\} \arrow[ur, bend left, "f"] \arrow[r, dotted, "(f,g)"] \arrow[dr, bend right, "g"'] \arrow[ur, phantom, shift right=3ex, "="] \arrow[dr, phantom, shift left=3ex, "="] & \{A \times B : \chi \wedge \varphi\} \arrow[u, "l"'] \arrow[d, "r"] \\
 & \{B : \varphi\}
\end{tikzcd}$$

$$\begin{aligned}
\mathbf{PR} \vdash \psi(c) &\implies \chi\, f(c) \wedge \varphi\, g(c) \\
&\iff [\chi \wedge \varphi]\, (f, g)(c) \text{ q. e. d.}
\end{aligned}$$

Assertion (iv) on object $\mathbb{2}$: it inherits its sum property $\mathbb{2} \cong \mathbb{1} + \mathbb{1}$

from $\mathbb{N} \cong \mathbb{1} + \mathbb{N}$, $b_0, b_1 : \mathbb{1} \to B$ already in **PRa**. Diagram:

$$\begin{tikzcd}
\mathbb{1} \arrow[d, "\mathrm{false}"', "0"] \arrow[drr, bend left, "b_0"] \arrow[dr, phantom, "="] & & \\
\mathbb{2} = \{\mathbb{N} : n < 2\} \arrow[rr, dotted, "b", "(b_0|b_1)"'] & & B \\
\mathbb{1} \arrow[u, "\mathrm{true}", "1"'] \arrow[urr, bend right, "b_1"'] \arrow[ur, phantom, "="] & &
\end{tikzcd}$$

PR map

$$b =_{\mathrm{def}} \mathrm{pr}[b_0, b_1 \circ r_{\mathbb{1},\mathbb{N}} \circ (\Pi, \mathrm{id})] :$$
$$\mathbb{N} \to \mathbb{1} \times \mathbb{N} \to B$$

does the job, uniquely with respect to equality of **PRa**, since with general parameter object A in **PRa** in place of $\mathbb{1} : A \times \mathbb{2} \cong A + A$, DIAGRAM:

$$\begin{tikzcd}
A \arrow[d, "(a{,}\mathrm{false})"'] \arrow[drr, bend left, "f"] \arrow[dr, phantom, "="] & & \\
A \times \mathbb{2} = A \times \{\mathbb{N} : n < 2\} \arrow[rr, dotted, "h", "(f|g)"'] & & B \\
A \arrow[u, "(a{,}\mathrm{true})"] \arrow[urr, bend right, "g"'] \arrow[ur, phantom, "="] & &
\end{tikzcd}$$

PRa map

$$h =_{\mathrm{def}} \mathrm{pr}[f, g \circ r_{A,\mathbb{N}} : (A \times \mathbb{N}_{<2}) \times B \xrightarrow{\ell\ell} A \xrightarrow{g} B] :$$
$$A \times \mathbb{2} \to B$$

does the job, uniquely with respect to equality of **PRa**, details as an **exercise.**

cartesian embedding assertion (v) is now obvious by construction of **PRa** over **PR**.

Proof of (vi) and (vii) are left as categorical **exercises** on construction of all finite *limits* out of *Extensions* of predicates, in particular on consatruction of pullbacks.

Proof of critical iteration assertion (viii): consider an endomorphism $f : \{A : \chi\} \to \{A : \chi\}$, so

$$\begin{aligned}&\mathbf{PR} \vdash \ \chi \implies \chi\, f : \\ &\quad A \xrightarrow{(\chi, \chi\, f)} \mathbb{N} \times \mathbb{N} \xrightarrow{\Rightarrow} \mathbb{N}.\end{aligned}$$

The iterated is the restriction of **PR** iterated $f^{\S} : A \times \mathbb{N} \to A$. Is it a **PRa** map $f^{\S} : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}\} \to \{A : \chi\}$?

Apply Peano Induction P5 (within **PR**) to predicate

$$\begin{aligned}&\varphi = \varphi(a, n) \ =_{\mathrm{def}} \ [\chi(a) \ \Rightarrow \ \chi\, f^n(a)] : A \times \mathbb{N} \to \mathbb{N}. \\ &\varphi(a, 0) = \mathrm{true} \text{ by anchoring } f^{\S}. \\ &[\varphi(a, n) \ \Rightarrow \ \varphi(a, s\, n)] \\ &\ = [[\chi(a) \ \Rightarrow \ \chi\, f^{\S}(a, n)] \ \Rightarrow \ [\chi(a) \ \Rightarrow \ \chi\, f^{\S}(a, s\, n)]] \\ &\ = [[\chi(a) \ \Rightarrow \ \chi\, f^{\S}(a, n)] \ \Rightarrow \ [\chi(a) \ \Rightarrow \ \chi\, f\, f^{\S}(a, n)]] \\ &\ = \mathrm{true},\end{aligned}$$

the latter by $f : \{A : \chi\} \to \{A : \chi\}$ a **PRa** map:

$$\mathbf{PR} \vdash \ \chi\, f^{\S}(a, n) \implies \chi\, f\, f^{\S}(a, n),$$

and by boolean tautology.

Peano Induction then gives $\varphi = \varphi(a,n) = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$, i. e. $f^\S : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}\} \to \{A : \chi\}$ is in fact a **PRa** map.

Compatibility of iteration with **PRa**'s equality: for endo maps $f =^{\mathbf{PRa}} g : \{A : \chi\} \to \{A : \chi\}$, i. e.

$$\mathbf{PR} \vdash\ \chi(a) \implies f(a) \doteq g(a) : A \to \mathbb{N},$$

we show

$$\mathbf{PR} \vdash\ \chi(a) \implies f^\S(a,n) \doteq g^\S(a,n) : A \times \mathbb{N} \to \mathbb{N}$$

by Peano Induction on

$$\varphi(a,n) = [\chi(a) \Rightarrow f^\S(a,n) \doteq g^\S(a,n)]$$

as follows:

anchor $\varphi(a,0) = \mathrm{true}_A$ is trivial. Step is an analogon to step above:

$$\begin{aligned}
&[\varphi(a,n) \Rightarrow \varphi(a,s\,n)] \\
&= [[\chi(a) \Rightarrow f^\S(a,n) \doteq g^\S(a,n)] \\
&\qquad \Rightarrow [\chi(a) \Rightarrow f^\S(a,s\,n) \doteq g^\S(a,s\,n)]] \\
&= [[\chi(a) \Rightarrow f^\S(a,n) \doteq g^\S(a,n)] \\
&\qquad \Rightarrow [\chi(a) \Rightarrow f\,f^\S(a,n) \doteq g\,g^\S(a,n)]] \\
&= \mathrm{true},
\end{aligned}$$

by $f =^{\mathbf{PRa}} g : \{A : \chi\} \to \{A : \chi\}$.

Peano Induction then gives $\varphi = \varphi(a,n) = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$, i. e. in fact

$$f^\S =^{\mathbf{PRa}} g^\S : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}\} \to \{A : \chi\}.$$

q. e. d.

Remark: In parallel to the above, Reiter 1980 shows that arithmetical theories allow for a formal extension by *quotient* sets A/ρ, $\rho : A \times A \to \mathbb{2}$ an *equivalence predicate* on A, the resulting theory being arithmetical again, and having extensions of predicates if this is the case for the original theory.

In fact, already $\mathbf{PRa} = \mathbf{PR} +$ (abstr) has these quotients, in form $A/\rho \; =_{\mathrm{def}} \; \{a \in A : a \doteq_A \bar{a}\}$ where $\bar{a} \; =_{\mathrm{def}} \; \min\{\tilde{a} \leq_A a : \tilde{a}\,\rho\,a\}$ is the minimal representant of the equivalence class of a, minimal with respect to the linear well order $\leq_A$: $A \times A \to \mathbb{2}$ on A which is given by Cantor's isomorphism $\mathrm{ct}_A : \mathbb{N} \xrightarrow{\cong} A$, A a nested binary power of $\mathbb{N}$, and its codomain restriction to subobjects $A' = \{A : \chi\}$ in **PRa**. In formal terms:

PRa admits the following scheme of forming Quotients by equivalence predicates:

$$\text{(QuotPred)} \quad \frac{\rho : \{A : \chi\}^2 \to \mathbb{2} \;\; \text{an equivalence predicate in } \mathbf{PRa}}{\begin{array}{l} [a]_\rho \; =_{\mathrm{def}} \; \min\{\tilde{a} \leq_A a : \tilde{a}\,\rho\,a\} : A \to A, \\ \{A : \chi\}/\rho \; =_{\mathrm{def}} \; \{a \in \{A|\chi\} : a \doteq_A [a]_\rho\}, \\ \quad \text{together with } \textit{quotient map} \\ \mathrm{nat}_\rho = \mathrm{nat}_\rho(a) \; =_{\mathrm{def}} \; [a]_\rho : \{A : \chi\} \to \{A : \chi\}/\rho \end{array}}$$

has the universal properties of a coequaliser of **PRa** pair
$\{(a', a'') \in \{A : \chi\}^2 : a'\,\rho\,a''\} \xrightarrow{\subseteq} A \times A \overset{l}{\underset{r}{\rightrightarrows}} A.$

(Here $[a]_\rho : \{A : \chi\} \to \{A : \chi\}$ is the *minimal representant* of the equivalence class $[a]_\rho$ of a.)

Map pair above is the canonical *kernel pair* $\mathrm{KP}[\mathrm{nat}_\rho]$ of quotient $\mathrm{nat}_\rho : \{A : \chi\} \twoheadrightarrow \{A : \chi\}/\rho$.

Remark: Generation of an *equivalence* out of an $\{A : \chi\}$ *predicate* $\sigma : A^2 \to \mathbb{2}$ gives in general only an equivalence *relation* $\overline{\sigma} : D_{\overline{\sigma}} \to A^2$, *not* a generated equivalence *predicate* $\overline{\sigma} : A^2 \to \mathbb{2}$: a priori we have no *decision,* if pairs in A^2 admit a joining σ-*transitivity chain.* These chains are p. r. *enumerable,* and this enumeration gives an *enumeration* of the relational *transitive hull* of a given *predicate,* and also that of a given *relation,* within domain A^2, A object of theory **PRa** or of a strengthening **S** of **PRa**.

The **Problem** of integrating *constructively* quotients by equivalence *relations* into a p. r. theory, is somewhat involved:

Reiter has formally added such quotients to cartesian p. r. theories with predicate abstraction (and before quotients by equivalence *predicates*), and obtains a cartesian p. r. theory with the original one embedded, preserving its structure, and gets this way a theory **Q** which has in addition quotients by those equivalence relations which are brought in by the original theory.

Iterating externally (!) this *stepwise closure* by quotients of equivalence relations, one arrives at a certain Closure of e.g. theory **PRa** under some important structural requirements:

This closure $\mathbf{S\bar{Q}}$ is a cartesian p. r. theory, it has sums, and quotients by equivalence relations, as well as—**Conjecture** at the moment—the usual, enumerative construction of quotients by arbitrary relations.

On the *projective-limit* side we will get $\mathbb{1}$ as the *terminal object,* and *cartesian product.* As far as I can see now, we will have equality predicates, and then equalisers a priori only of maps out of theories **PRa** and strengthenings **S**, but nevertheless with their *universal properties* preserved by the embedding into the hull.

For a long while we will need just quotients by those equivalence relations which come in from theory **PRa** resp. its strengthenings, **S** in general. We call this REITER's theory **PRaQ** = **PRa** + Quot, **SQ** = **S** + Quot, in general.

For the remainder of this book, let **S** be theory **PRa** or a *strengthening* of **PRa** by additional axioms.

What we need here is that such a theory **S** has extensions of all of its predicates, as well as limits of finite **S**-Diagrams, in particular **S** *pullbacks.* And that it admits, as a cartesian theory, canonical interpretation of free variables as identities resp. possibly nested projections.

As a fundamental theorem for such Extensions **S** of **PR**, we get the following scheme:

Equality definability theorem: Theory **S** admits the following scheme:

$$
\text{(EquDef)}\quad
\frac{\begin{array}{l} f, g : A \to \mathbb{N} \text{ in } \mathbf{S}, \\ \mathbf{S} \vdash \ [f(a) \doteq g(a)] : A \to \mathbb{N}^2 \to \mathbb{2} \end{array}}{\begin{array}{l} \mathbf{S} \vdash \ f = g : A \to \mathbb{N}, \ \textit{algebraically:} \\ f =^{\mathbf{S}} g : A \to \mathbb{N}. \end{array}}
$$

Equality definability extends to **S**-map pairs $f, g : A \to B$ with common codomain a cartesian product B or even B an object of theory **PRa**.

Proof by commutativity $\max(m,n) = \max(n,m) : \mathbb{N}^2 \to \mathbb{N}$, cf. corressponding result for Goodstein Arithmetic **GA** :

Equality *predicate* $\doteq$ on $\mathbb{N}$ has been defined earlier as

$$[\,m \doteq n\,] \ =_{\mathrm{def}} \ \neg\,|m - n| \ =_{\mathrm{by\,def}} \ \neg[(m \dot{-} n) + (n \dot{-} m)] : \mathbb{N}^2 \to \mathbb{2},$$

using *truncated subtraction* $(m \dot{-} n) : \mathbb{N}^2 \to \mathbb{N}$, and negation $\neg(a) : \mathbb{N} \to \mathbb{2}$.

Substitution (realised as composition), of $f : A \to \mathbb{N}$ into m, and $g : A \to \mathbb{N}$ into n gives:

$$[\,f \doteq g\,] \ =_{\mathrm{by\,def}} \doteq \circ\,(f, g) =^{\mathbf{S}} \neg\, \circ\, |f - g| : A \to \mathbb{N}^2 \to \mathbb{N} \to \mathbb{N},$$

and further, with commutativity of the max function[2] namely

$$\max(m,n) \ =_{\mathrm{by\,def}} \ m + (n \dot{-} m) = n + (m \dot{-} n) \ =_{\mathrm{by\,def}} \ \max(b,a),$$

[2]see section on Goodstein Arithmetic above

$$\frac{\begin{array}{l} f, g : A \to \mathbb{N} \text{ in } \mathbf{S}, \\ \mathrm{true}_A = [\, f \doteq g\,] : A \to \mathbb{N}^2 \to \mathbb{2} \end{array}}{\begin{array}{l} |f - g| = 0 = |g - f|, \text{ hence} \\ (f \dot{-} g) = 0 = (g \dot{-} f), \text{ and hence} \\ f = f + (g \dot{-} f) \ =_{\text{by def}} \ \max(f, g) \\ = \max(g, f) \ =_{\text{by def}} \ g + (f \dot{-} g) \\ = g : A \to \mathbb{N}. \end{array}}$$

The case of B an arbitrary **PRa**-object follows from the above by definition of equality predicate $[\, b \doteq_B b'\,] : B^2 \to \mathbb{2}$ via *conjunction* of equality predicates on the *components* and *restriction* to predicate extensions $\{C : \varphi\}$ **q. e. d.**

Leibniz substitutivity for this (family of) equality *predicates* $\doteq_A : A^2 \to \mathbb{2}$ reads:

$$(Sub_{\doteq}) \quad \frac{f : A \to B \text{ in } \mathbf{S} \text{ i. e. in } \mathbf{PRa}}{\mathbf{S} \vdash \ [\, a \doteq a'\,] \implies [\, f(a) \doteq_B f(a')\,]}$$

Proof see earlier Proof of *Leibniz substitutivity theorem* for theory **PR**, easily extended to present strengthening **S** of theory **PRa** $=$ **PR** $+$ (abstr).

Bottom up resolution lemma for primitive recursion: For any p. r. theory $\mathbf{T}$, and a $\mathbf{T}$ endo map $f : A \to A$ we have iterated $f^{\S} : A \times \mathbb{N} \to A$ characterised by

$$f^{\S}(a, 0) = a = \mathrm{id}_A : A \to A \quad \text{(anchor)},$$
as before, and
$$f^{\S}(a, s\,n) = f^{\S}(f(a), n). \quad \text{(step)}_{\text{bottomup}}$$

Remarks:

- a **PRa**-map $f : \{A : \chi\} \to \{B : \varphi\}$ can be viewed as a *defined partial* **PR** map from A to B with values in φ : the object of *defined arguments,* namely $\{a \in A : \chi(a)\}$ is p. r. *decidable.* By definition of **PRa**'s equality, **PR**-map $f : A \to B$ "doesn't care" about arguments a in the *complement* $\{a \in A : \neg\,\chi(a)\}$.

  So wouldn't it be easier to realise this view to *defined partial maps* just by throwing the *undefined arguments* into a *waste basket* $\{\bot\}$?

  But where to place this waste basket, this for each codomain object B? The fundamental objects have a zero-vector as a candidate. For example we could interprete truncated subtraction as a *defined partial* map

  $$a \dot{-} b : \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \geq n\} \to \mathbb{N},$$

  and throw the complement $\{(m, n) \in \mathbb{N} \times \mathbb{N} : m < n\}$ into waste basket $\{0\} \subset \mathbb{N}$. But this is not a good interpretation of *truncated* (!) subtraction: Value 0 is *not* waste, it has an important meaning as zero.

"The" waste basket $\{\bot\}$ should be an entity with a *natural* extra representation, and we should have only one such entity in a later theory of defined partial p. r. maps to come. This theory, to be called $\mathbf{PR}\mathbb{X}\mathbf{a}$, will be constructed with the help of a *universal object* $\mathbb{X}$ which is to contain all *numerals* (codes of numbers) and all nested pairs of numerals. It then has place for LaTeXcodes of all symbols, in particular for the code $\underline{\bot}$ of *undefined value* symbol $\underline{\bot}$, in a "Hilbert's hotel".

- a **PR**-map $f : A \to B$ such that f is a **PRa**-map

  $f : \{A : \chi \vee \chi' : A \to \mathbb{2}\} \to \{B : \varphi\}$
  also works as a **PRa**-map
  $f : \{A : \chi\} \to \{B : \varphi\},$ and a **PRa**-map
  $g : \{A : \chi\} \to \{B : \varphi \wedge \varphi'\}$
  also works as a **PRa**-map
  $g : \{A : \chi\} \to \{B : \varphi\}.$

Since map-properties of *injectivity, epi-property* of **PR**-maps viewed as **PRa**-maps depend on choice of hosting **PRa** objects—**examples** above—*specification* of a **PRa** map $f : \{A : \chi\} \to \{B : \varphi\}$ must contain, besides **PR**-map $f : A \to B$, domain and codomain *objects* $\chi : A \to \mathbb{2}$ and $\varphi : B \to \mathbb{2}$ as well.

This way the members of map set family $\mathbf{PRa}(A, B) :\ A, B$ **PRa**-objects, become mutually disjoint. Inclusions $i : A' \xrightarrow{\subseteq} A''$ are realised in **PRa** as restricted **PR**-identities

$\mathrm{id}_A : \{A : \chi'\} \xrightarrow{\subseteq} \{A : \chi''\},\ \chi' \implies \chi''.$

2.2 Predicate calculus

Free Variables Predicate Calculus

In the framework **GA** $\subseteq$ **PR** $\sqsubset$ **PRa** of **Goodstein Arithmetic** we have introduced on NNO $\mathbb{N}$ the following proto Boolean structure:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "\mathrm{false}\equiv 0"] \arrow[r, shift right, "\mathrm{true}\equiv 1"'] & \mathbb{N} \arrow[d, bend right=50, "\neg"'] \arrow[d, ">0"', "\mathrm{sign}"] & \mathbb{N}\times\mathbb{N} \arrow[l, shift right, "\vee\equiv +"'] \arrow[l, shift left, "\wedge\equiv\cdot"'] \arrow[dl, shift right, "\Rightarrow\,\equiv\leq"] \arrow[dl, shift left, "\Leftrightarrow\equiv\doteq"'] \\
& \mathbb{N} &
\end{tikzcd}$$

This structure is turned, within **PRa**, into a two-valued Boolean algebra on object

$$\begin{aligned}
\mathbb{2} \ &=_{\text{by def}} \ \{0,1\} \\
&=_{\text{def}} \ \{n \in \mathbb{N} : n \doteq 0 \ \vee \ n \doteq 1\} \\
&=_{\text{by def}} \ \{n \in \mathbb{N} : n \leq 1\} :
\end{aligned}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "\mathrm{false}\equiv 0"] \arrow[r, shift right, "\mathrm{true}\equiv 1"'] & \mathbb{2} \arrow[d, bend right=40, "\neg"'] \arrow[d, "\mathrm{id}"] & \mathbb{2}\times\mathbb{2} \arrow[l, shift right, "\vee"'] \arrow[l, shift left, "\wedge"'] \arrow[dl, shift right, "\Rightarrow\,\equiv\leq"] \arrow[dl, shift left, "\Leftrightarrow\equiv\doteq"'] \\
& \mathbb{2} &
\end{tikzcd}$$

A **PR** *predicate* on an object A of **PR** has been a **PR** map $\chi : A \to \mathbb{N}$ with

$\mathrm{sign} \circ \chi = \chi,$

$$\begin{tikzcd} A \arrow[r, "\chi"] \arrow[rr, bend right, "\chi"'] & \mathbb{N} \arrow[r, "\mathrm{sign}"] \arrow[d, phantom, "="] & \mathbb{N} \\ & {} & \end{tikzcd}$$

A **PRa** predicate on an object $\{A : \chi\}$ is a **PRa** map $\varphi = \varphi(a) : \{A : \chi\} \to \mathbb{2} = \{0, 1\}$.

Using the Boolean operations on $\mathbb{2}$ above, a *free-Variables boolean predicate calculus* is easily defined, making the set of **PR** predicates on (any) object A of **PRa** into a boolean algebra:

- overall negation:

  $\neg\,\varphi(a) = \neg \circ \varphi : A \to \mathbb{2} \to \mathbb{2},$

- conjunction:

  $\chi(a) \wedge \varphi(a) = \wedge \circ (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- disjunction:

  $\chi(a) \vee \varphi(a) = \vee \circ (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- implication:

  $[\chi(a) \;\Rightarrow\; \varphi(a)] = \;\Rightarrow\; \circ\, (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- equivalence:

  $[\chi(a) \Leftrightarrow \varphi(a)] = \doteq_{\mathbb{2}} \circ\, (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

Verification of the logical properties of such free-variables predicates and their interrelationships by the *truth table method* inherited from the Boolean algebra $\mathbb{2}$.

Axiomatic Images and Quantification

As a step aside, we discuss here classical quantification, introduced axiomatically via image predicates. These correspond to topos theoretic characteristic functions of non-necessarily monic (injective) maps. quantification + cartesian PR allows for the original version of Gödel's theorems. It seems to be necessary for that original theorems and proof, since existential quantification plays a prominent rôle in statement and proof. Nevertheless, Incompleteness can be shown in a different way for weaker theories, cf. GOODSTEIN 1957. We do not exclude that $\mathbf{PR}, \mathbf{PRa}$ turn out to be incomplete in Goodstein's sense.

Definition: A (total) predicate $\chi : B \to \mathbb{2}$ is a (the) *image predicate* of a map $f = f(a) : A \to B$, if

- $\chi \circ f = \mathrm{true}_A : A \to B \to \mathbb{2}$ and
- $\chi : B \to \mathbb{2}$ *minimal* in this regard i. e.

$$\frac{\varphi \circ f = \mathrm{true}_A : A \to B \to \mathbb{2}}{[\chi(b) \;\Rightarrow\; \varphi(b)] = \mathrm{true}_B}$$

If available, such χ, noted $\mathrm{im}[f] = \mathrm{im}[f](b) : B \to \mathbb{2}$, is unique, this by minimality and Equality Definability.

In case of $f : A \to B$ monic, such χ is just the characteristic map of f in the sense of Elementary Topos theory **ETT,** with respect to $\mathbb{2} = \{0, 1\} \subset \mathbb{N}$ taken as its *subobject classifier, truth object.*

If available, image

$$\mathrm{im}[\{A \times B : \varphi\} \xrightarrow{\subseteq} A \times B \xrightarrow{\ell} A] : A \to \mathbb{2}$$

works as *right existential quantification*

$$(\exists b \in B)\varphi(a, b) = (\exists_r \varphi)(a) : A \to \mathbb{2},$$

with the categorical properties of this quantification known from (**ETT** and categorical) **set** theory.

If available, define *right universal quantification*

$$(\forall b \in B)\varphi(a, b) \ =_{\text{def}} \ \neg\,(\exists b \in B)\,\neg\varphi(a, b) : A \to \mathbb{2}.$$

Our (weak, categorical) set theories $\mathbf{T}$ will here always be Extensions of quantified p. r. theory $\mathbf{PRa}\exists = \mathbf{PRa} + (\exists)$, **defined** to be theory **PRa** closed under formation of images and hence closed under (two-valued) quantification $\exists, \forall$.

Comment: These *semi-classical* theories will be taken as *background* for Consistency questions: we will show differences in internal consistency between these classical set theories $\mathbf{T}$, in particular between Osius' categorical pendants of the different stages of Zermelo-Fraenkel set theory **ZF** on one hand, and the categorical theories here: $\mathbf{PR}, \mathbf{PRa}$ above, and $\mathbf{PR}\mathbb{X}, \mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R}$ to come. For fixing ideas, you may always read set theory $\mathbf{T}$ as $\mathbf{T} := \mathbf{PRa}\exists$: Gödel's Incompleteness theorems apply to $\mathbf{PRa}\exists$, not to *descent* p. r. theory $\pi\mathbf{R}$ to come.

Notes

(a) we have equalisers, products distributing over sums, sums certainly stable under pullbacks, quotients by equivalence predicates (not yet quotients by equivalence relations).

(b) in comparison with doctrines: Kock-Reyes 1977, and in comparison with pretopoi: Maietti 2010(?), (axiomatic) quantification is lacking for “our” strengthenings **S** of **PRa**.

Chapter 3

Partial Maps

We introduce μ-recursive maps as *partial p. r. maps,* coming as a p. r. enumeration of *defined arguments* together with a p. r. *rule* mapping the enumeration index of a defined argument into the *value* of that argument. This covers μ-recursive maps and content driven loops as in particular while-loops. Code evaluation will be definable as such a while-loop. Only these *concepts* and **section lemma** in **structure theorem** for $\mathbf{P\widehat{R}a}$ will be used in the subsequent chapters on decidability and consistence.

Present chapter constitutes a structure theory on its own, it reflects Budach & Hoehnke 1975.

3.1 Theory of partial maps

Definition: A partial map $f : A \rightharpoonup B$ is a pair

$$f = \langle d_f : D_f \to A,\ \widehat{f} : D_f \to B \rangle : A \rightharpoonup B,$$

$$\begin{tikzcd}
D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & \\
A \arrow[r, harpoon, "f"'] & B
\end{tikzcd}$$

The pair $f = \langle d_f \,, \widehat{f} \rangle$ is to fullfill the **right-uniqueness condition**

$$d_f(\hat{a}) \doteq_A d_f(\hat{a}') \implies \widehat{f}(\hat{a}) \doteq_B \widehat{f}(\hat{a}') :$$

We now define the theory $\widehat{\mathbf{S}}$ of *partial* $\mathbf{S}$-maps $f : A \rightharpoonup B$.

Objects of $\widehat{\mathbf{S}}$ are those of $\mathbf{S}$, i. e. of $\mathbf{PRa}$. The *morphisms* of $\widehat{\mathbf{S}}$ are the *partial* $\mathbf{S}$-maps $f : A \rightharpoonup B$.

Definition: Given $f', f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, we say that f *extends* f' or that f' is a *restriction* of f, written $f' \,\widehat{\subseteq}\, f$, if there is given a map $i : D_{f'} \to D_f$ in $\mathbf{S}$ such that

$$(f' \widehat{\subseteq} f) \qquad \begin{tikzcd}
 & D_{f'} \arrow[d, "i"] \arrow[ddl, bend right, "d_{f'}"'] \arrow[ddr, bend left, "\widehat{f'}"] & \\
\arrow[r, phantom, "=^{\mathbf{S}}"] & D_f \arrow[dl, "d_f"'] \arrow[dr, "\widehat{f}"] \arrow[r, phantom, "=^{\mathbf{S}}"] & \, \\
A \arrow[rr, harpoon, "f'", "f"'] & & B
\end{tikzcd}$$

The partial maps f and f' are *equal* in $\widehat{\mathbf{S}}$, if f extends f' and f'

extends f :

$$(\,\widehat{=}^{\,\mathbf{S}})\quad \frac{f' \,\widehat{\subseteq}\, f,\ \ f \,\widehat{\subseteq}\, f' : A \rightharpoonup B}{f' \,\widehat{=}\, f : A \rightharpoonup B.}$$

Notation: From now on, $f = g : A \to B$ will always denote equality between maps within theory $\mathbf{S}$ choosen as *basic,* cartesian p. r. theory. Equality between *partial* $\mathbf{S}$-maps, $\widehat{\mathbf{S}}$-morphisms $f,\ g : A \rightharpoonup B$ is denoted $f \,\widehat{=}\, g : A \rightharpoonup B$, see the above. Pointed equality $\doteq\, : \mathbb{N}^2 \to \mathbb{2}$ resp. $\doteq_A : A^2 \to \mathbb{2}$ is reserved for equality *predicates* (special *maps*), *on* $\mathbb{N}$ resp. *on* objects A of $\mathbf{S}$.

Definition: *Composition* $h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$ of $\widehat{\mathbf{S}}$ maps

$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$ and

$g = \langle (d_g, \widehat{g}) : D_g \to B \times C \rangle : B \rightharpoonup C$

is **defined** by the diagram

$$\begin{tikzcd}
D_h \arrow[d, "\pi_l"] \arrow[dr, "\pi_r"] \arrow[ddrr, bend left, "\widehat{h}"] \arrow[dd, bend right=60, "d_h"'] & & \\
D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] \arrow[r, phantom, "\text{p.b.}"] & D_g \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] \arrow[r, phantom, "="] & {} \\
A \arrow[r, harpoon, "f"] \arrow[rr, harpoon, bend right, "\widehat{=}", "h = g \,\widehat{\circ}\, f"'] & B \arrow[r, harpoon, "g"] & C
\end{tikzcd}$$

Composition DIAGRAM for $\widehat{\mathbf{S}}$

[The idea is from Brinkmann-Puppe 1969: They construct composition of *relations* this way via pullback]

Remark: The *standard form* of the pullback D_h is

$$D_h = \{(\hat{a}, \hat{b}) \in D_f \times D_g : \widehat{f}(\hat{a}) \doteq_B d_g(\hat{b})\},$$

with pullback-*projections*

$$l = \pi_l = l \circ \subseteq : D_h \to D_f \times D_g \to D_f \text{ and}$$
$$r = \pi_r = r \circ \subseteq : D_h \to D_f \times D_g \to D_g.$$

[We may abbreviate such *restricted* projections—*pullback "projections"*—π_l and π_r respectively, by l, r—as suggested above]

In a sense, the pullback D_h represents the inverse image $D_h = \overset{-1}{f}[D_g]$, more precisely: $[D_h \xrightarrow{l} D_f] = \overset{-1}{\widehat{f}}[D_g \xrightarrow{d_g} B]$. But the definability domains d_f, d_g, d_h need not be monic (injective).

Composition $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$ gives a *well-defined* partial map h, since for $(\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h$ free:

$$\begin{aligned}
&d_h(\hat{a}, \hat{b}) \doteq_A d_h(\hat{a}', \hat{b}') \iff d_f(\hat{a}) \doteq_A d_f(\hat{a}') \\
&\implies \widehat{f}(\hat{a}) \doteq_B \widehat{f}(\hat{a}') \ (f \text{ well-defined}), \\
&\iff \widehat{f}\, l(\hat{a}, \hat{b}) \doteq \widehat{f}\, l(\hat{a}', \hat{b}') \\
&\implies d_g(r(\hat{a}, \hat{b})) \doteq_B d_g(r(\hat{a}', \hat{b}')) \\
&\qquad (\ (\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h, \text{ p.b. commutes}) \\
&\iff d_g(\hat{b}) \doteq_B d_g(\hat{b}') \implies \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') \\
&\implies \widehat{h}(\hat{a}, \hat{b}) = \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') = \widehat{h}(\hat{a}', \hat{b}') : D_h \times D_h \to 2.
\end{aligned}$$

Obviously, $\widehat{\mathbf{S}}$-map $\mathrm{id}_A^{\widehat{\mathbf{S}}} =_{\mathrm{def}} \langle(\mathrm{id}_A, \mathrm{id}_A) : A \to A^2\rangle : A \rightharpoonup A$ works as *identity* for object A with respect to composition $\widehat{\circ}$ for $\widehat{\mathbf{S}}$.

If one of two $\widehat{\mathbf{S}}$ maps to be composed, is an $\mathbf{S}$ map, $\widehat{\mathbf{S}}$ *composition* becomes simpler:

Mixed Composition Lemma:

(i) For $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, and $g : B \to C$ in $\mathbf{S}$:

$$g \,\widehat{\circ}\, f = \langle(d_f, g \circ \widehat{f}) : D_f \to A \times C\rangle : A \rightharpoonup C,$$

in DIAGRAM form:

$$\begin{tikzcd}
D_f \arrow[d, "d_f"'] \arrow[dr, "\widehat{f}"'] \arrow[drr, bend left, "g \circ \widehat{f}"] & & \\
A \arrow[r, harpoon, "f"] \arrow[rr, bend right, harpoon, "g \,\widehat{\circ}\, f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

(ii) For $f : A \to B$ in $\mathbf{S}$, $g : B \rightharpoonup C$ in $\widehat{\mathbf{S}}$:

$$g \,\widehat{\circ}\, f = \langle(\overset{-1}{f}[d_g], \widehat{g} \circ \bar{f}) : \overset{-1}{f}[D_g] \to A \times C\rangle : A \rightharpoonup C,$$

as DIAGRAM:

$$\begin{tikzcd}
\overset{-1}{f}[D_g] \arrow[r, "\bar{f}"] \arrow[d, "\overset{-1}{f}[d_g]"'] \arrow[dr, phantom, "\text{p.b.}"] & D_g \arrow[d, "d_g"'] \arrow[dr, "\widehat{g}"] & \\
A \arrow[r, "f"] \arrow[rr, bend right, harpoon, "g \,\widehat{\circ}\, f"'] & B \arrow[r, harpoon, "g"] & C
\end{tikzcd}$$

Proof: Left as an **exercise.**

3.2 Structure theorem for $\mathbf{P\widehat{R}a}$:

(i) $\widehat{\mathbf{S}}$ carries a canonical structure of a *diagonal symmetric monoidal category*, with composition $\widehat{\circ}$ and identities introduced above, monoidal product $\times$ extending $\times$ of $\mathbf{S}$, *association* $\mathrm{ass} : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$, *symmetry* $\Theta : A \times B \xrightarrow{\cong} B \times A$, and *diagonal* $\Delta : A \to A \times A$ inherited from $\mathbf{S}$.

(ii) The defining diagram for a $\widehat{\mathbf{S}}$-map—namely

$$\begin{tikzcd}
D_f \arrow[d, "d_f"] \arrow[dr, bend left, "\widehat{f}"] & \\
A \arrow[r, rightharpoondown, "f"] & B
\end{tikzcd}$$

Partial Map DIAGRAM

is a commuting $\widehat{\mathbf{S}}$ diagram.

Conversely the *minimised opposite* $\widehat{\mathbf{S}}$ map $d_f^- : A \rightharpoondown D_f$ to $\mathbf{S}$ map $d_f : D_f \to A$ fullfills

$$\begin{tikzcd}
D_f \arrow[dr, bend left, "\widehat{f}"] \arrow[dr, phantom, "\widehat{=}"] & \\
A \arrow[u, "d_f^-"] \arrow[r, rightharpoondown, "f"] & B
\end{tikzcd}$$

Put together:

$$\begin{tikzcd} D_f \arrow[d, shift right, "d_f"'] \arrow[dr, bend left, "\widehat{f}"] \arrow[dr, phantom, "\widehat{=}"] & \\ A \arrow[u, shift right, "d_f^-"'] \arrow[r, harpoon, "f"] & B \end{tikzcd}$$

basic partial map DIAGRAM

(iii) **"section lemma:"** The first factor $f : A \rightharpoonup B$ in an $\widehat{\mathbf{S}}$ composition

$$h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C,$$

when giving an (embedded) $\mathbf{S}$ map $h : A \to C$, is itself an (embedded) $\mathbf{S}$ map:

a first composition factor of a total map is total.

So each section ("coretraction") of theory $\widehat{\mathbf{S}}$ is an $\mathbf{S}$ map, in particular an $\widehat{\mathbf{S}}$ section of an $\mathbf{S}$ map belongs to $\mathbf{S}$.

(iv) $\widehat{\mathbf{S}}$ clearly inherits from $\mathbf{S}$ *surjective pairing* (SP):

For $h : C \rightharpoonup A \times B$ in $\widehat{\mathbf{S}}$,

$$h \mathrel{\widehat{=}} (h \mathbin{\widehat{\circ}} l, h \mathbin{\widehat{\circ}} r) : C \rightharpoonup A \times B,$$

where for $f : C \rightharpoonup A$, $g : C \rightharpoonup B$,

$$(f, g) \;=_{\mathrm{def}}\; (f \times g) \mathbin{\widehat{\circ}} \Delta_C : C \to C \times C \rightharpoonup A \times B,$$

with *diagonal* $\Delta_C : C \to C \times C$ of $\mathbf{S}$.

This equation guarantees *uniqueness* of the "*induced*" $(f, g) : C \rightharpoonup A \times B$, but (f, g) does not satisfy (both of) the *cartesian* *equations*

$$l \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} f \text{ and } r \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} g,$$

except f and g have *equal domains of definition*, i. e. if $i : D_f \to D_g$, $j : D_g \to D_f$ are available such that $d_g \circ\ i = d_f : D_f \to A$ as well as $d_f \circ\ j = d_g : D_g \to A$.

(v) *Iteration* $f^{\S} : A \times \mathbb{N} \rightharpoonup A$ of $\widehat{\mathbf{S}}$-endo is available in $\widehat{\mathbf{S}}$, satisfying the characteristic $\widehat{\mathbf{S}}$-equations

$$\begin{aligned} & f^{\S} \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \ =_{\text{by def}} \ f^{\S} \mathbin{\widehat{\circ}} (A \times 0) \circ \Delta_A \mathrel{\widehat{=}} \mathrm{id}_A : A \to A, \text{ and} \\ & f^{\S} \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} f \mathbin{\widehat{\circ}} f^{\S} : A \times \mathbb{N} \rightharpoonup A. \end{aligned}$$

(vi) **conjecture:** Freyd's uniqueness of the *initialised iterated* holds in $\widehat{\mathbf{S}}$:

$$(\mathrm{FR!})_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} f : A \rightharpoonup B,\ g : B \rightharpoonup B,\ h : A \times \mathbb{N} \rightharpoonup B \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ h \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \rightharpoonup f : A \rightharpoonup B \quad \text{and} \\ h \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} h : A \times \mathbb{N} \rightharpoonup B \end{array}}{h \mathrel{\widehat{=}} g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) : A \times \mathbb{N} \rightharpoonup B \times \mathbb{N} \rightharpoonup B.}$$

(vii) For extension $\widehat{\mathbf{S}}$ of $\mathbf{S}$, we get—by the general Freyd's argument—from these assertion (v) and conjecture (vi), the **full scheme of**

primitive recursion, existence resp. uniqueness:

$$(\mathrm{pr})_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}} \text{ (initialisation)}, \\ h : (A \times \mathbb{N}) \times B \rightharpoonup B \text{ (step map)} \end{array}}{\begin{array}{l} f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \rightharpoonup B \text{ is available in } \widehat{\mathbf{S}}, \\ \text{characterised (up to equality } \widehat{=}\text{) in } \widehat{\mathbf{S}} \text{ by} \\ f \,\widehat{\circ}\, (\mathrm{id}_A, 0) \,\widehat{=}\, g : A \rightharpoonup B \text{ and} \\ f \,\widehat{\circ}\, (A \times s) \,\widehat{=}\, h \,\widehat{\circ}\, (\mathrm{id}_{A \times \mathbb{N}}, f) \\ \;=_{\text{by def}}\; h \,\widehat{\circ}\, ((A \times \mathbb{N}) \times f) \,\widehat{\circ}\, \Delta_{A \times \mathbb{N}} : \\ A \times \mathbb{N} \to (A \times \mathbb{N})^2 \rightharpoonup (A \times \mathbb{N}) \times B \rightharpoonup B. \end{array}}$$

Remark: The iteration and recursion properties (v),(vi),(vii) of the theorem will not be used in the subsequent chapters.

Proof of assertion (i): We first give to $\mathbf{S}$ the structure of a diagonal monoidal category and verify the defining properties of this structure:

Composition $\widehat{\circ}$ introduced above—by pullback—is compatible with $\widehat{\subseteq}$, and hence also with $\widehat{=}$, since for $f' \,\widehat{\subseteq}\, f : A \rightharpoonup B$ and $g' \,\widehat{\subseteq}\, g : B \rightharpoonup C$, we are given "inclusions" $i : D_{f'} \to D_f$ and $j : D_{g'} \to D_g$ such that for $h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$ and $h' = g' \,\widehat{\circ}\, f' : A \rightharpoonup B \rightharpoonup C$ compatibility DIAGRAM below commutes, with (unique) $k : D_{h'} \to D_h$ in $\mathbf{S}$, *induced into the pullback* D_h by $i \circ l' : D_{h'} \to D_{f'} \to D_f$ and $j \circ r' : D_{h'} \to D_{g'} \to D_g$.

$$
\begin{tikzcd}
 & & D_{h'} \arrow[dl, "l'"'] \arrow[dr, "r'"] \arrow[ddll, bend right, "d_{h'}"'] \arrow[ddrr, bend left, "\widehat{h}'"] \arrow[dddd, dotted, bend right=90, looseness=1.5, "k"'] & & \\
 & D_{f'} \arrow[dl, "d_{f'}"'] \arrow[dr, "\widehat{f}'"] \arrow[dd, dotted, "i"] \arrow[rr, phantom, "\text{p.b.}"] & & D_{g'} \arrow[dl, "d_{g'}"'] \arrow[dr, "\widehat{g}'"] \arrow[dd, dotted, "j"] & \\
A \arrow[rr, "f", "f'"'] \arrow[rrrr, bend right=15, "h", "h'"'] & & B \arrow[rr, "g", "g'"'] & & C \\
 & D_f \arrow[ul, "d_f"] \arrow[ur, "\widehat{f}"'] \arrow[rr, phantom, "\text{p.b.}"] & & D_g \arrow[ul, "d_g"] \arrow[ur, "\widehat{g}"'] & \\
 & & D_h \arrow[ul, "l"] \arrow[ur, "r"'] \arrow[uull, bend left, "d_h"] \arrow[uurr, bend right, "\widehat{h}"'] & &
\end{tikzcd}
$$

Compatibility DIAGRAM[a] of $\widehat{\circ}$ with $\subseteq$

[a] F. Herrmann

For proving associativity of (partial) composition $\widehat{\circ}$, consider

$$
\begin{tikzcd}
D_{(h\widehat{\circ}g)\widehat{\circ}f} \arrow[dr, leftrightarrow, "\cong"] \arrow[drr] \arrow[dddr] & & & & \\
 & D_{h\widehat{\circ}(g\widehat{\circ}f)} \arrow[d] \arrow[ddrr] \arrow[r, phantom, "\text{p.b.}" near start, shift left=4] & D_{h\widehat{\circ}g} \arrow[dd] \arrow[ddr] & & \\
 & D_{g\widehat{\circ}f} \arrow[d] \arrow[dr] \arrow[r, phantom, "\text{p.b.}"] & \arrow[r, phantom, "\text{p.b.}"] & \; & \\
 & D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] \arrow[r, phantom, "\text{p.b.}"] & D_g \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] & D_h \arrow[d, "d_h"] \arrow[dr, "\widehat{h}"] & \\
 & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & D
\end{tikzcd}
$$

Associativity DIAGRAM for $\widehat{\circ}$— via *nested pullbacks*

Here the standard form of isomorphism $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$ is restriction of *association isomorphism*

$$\mathrm{ass} : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$$

to a map $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$.

The (monoidal) product $f \times g : A \times B \rightharpoonup A' \times B'$ of partial maps is given *componentwise* as the *hook*

$$\begin{tikzcd}
D_f \times D_g \arrow[d, "d_f \times d_g"'] \arrow[dr, "\widehat{f} \times \widehat{g}"] & \\
A \times B \arrow[r, harpoon, "f \times g"'] & A' \times B'
\end{tikzcd}$$

In particular, cylindrification with an object A is the hook

$$\begin{tikzcd}
A \times D_g \arrow[d, "A \times d_g"'] \arrow[dr, "A \times \widehat{g}"] & \\
A \times B \arrow[r, harpoon, "A \times g"'] & A \times B'
\end{tikzcd}$$

Cylindrification preserves inclusion $f' \mathrel{\widehat{\subseteq}} f : A \rightharpoonup B$ given by $i : D'_f \to D_f$, since

$$C \times i : D_{C \times f'} = C \times D_{f'} \to C \times D_f = D_{C \times f}$$

gives the inclusion $C \times f' \mathrel{\widehat{\subseteq}} C \times f : C \times A \rightharpoonup C \times B$. Hence in particular, cylindrification preserves (partial) equality $f' \mathrel{\widehat{=}} f$ defined by $f' \mathrel{\widehat{\subseteq}} f$ and $f \mathrel{\widehat{\supseteq}} f'$ being given simultaneously.

As for **S**, the product of maps is given alternatively by composition of cylindrifications, namely

$$(\times_{\widehat{\mathbf{S}}}) \quad \frac{f : A \rightharpoonup A',\ g : B \rightharpoonup B' \text{ in } \widehat{\mathbf{S}}}{\begin{aligned}(f \times g) \ =_{\mathrm{def}}\ & (f \times B') \,\widehat{\circ}\, (A \times g) : \\ & A \times B \rightharpoonup A \times B' \rightharpoonup A' \times B' \\ \widehat{=}\ & (A' \times g) \,\widehat{\circ}\, (f \times B) : \\ & A \times B \rightharpoonup A' \times B \rightharpoonup A' \times B'.\end{aligned}}$$

It extends the *cartesian* product of **S** into a *bifunctor* again, on theory $\widehat{\mathbf{S}}$. Within $\widehat{\mathbf{S}}$, this product looses its universal property, essentially since already $[\,\Pi_A : A \to \mathbb{1}\,]_{A \in \mathbf{Obj_S}}$ looses *naturality,* within $\widehat{\mathbf{S}}$:

In general *domain of definition* $d_f : D_f \to A$ of a *partial* $f = (d_f, \widehat{f}) : A \rightharpoonup B$ does not cover the whole of *domain* A, whence

$$\begin{tikzcd} A \arrow[r, rightharpoonup, "f"] \arrow[d, "\Pi"'] \arrow[dr, phantom, "\not\widehat{=}"] & B \arrow[d, "\Pi"] \\ \mathbb{1} \arrow[r, equal] & \mathbb{1} \end{tikzcd}$$

Proof of bifunctoriality of $\times$ in $\widehat{\mathbf{S}}$:

The point here is functoriality of cylindrification, namely of

$$\langle g : B \to B' \rangle \mapsto \langle A \times g : A \times B \to A \times B' \rangle :$$

For partial maps $\langle (d_g, \widehat{g}) : D_g \to B \times B' \rangle : B \rightharpoonup B'$ and $\langle (d_{g'}, \widehat{g'}) : D'_g \to B' \times B'' \rangle : B' \rightharpoonup B''$, and a ("cylindrifying") object A, recall the following defining $\mathbf{S}/\widehat{\mathbf{S}}$ DIAGRAM for g, g', and $h := g' \,\widehat{\circ}\, g$:

$$
\begin{tikzcd}
D_h \arrow[d, "\pi_l"] \arrow[dr, "\pi_r"] \arrow[ddrr, bend left, "\widehat{h}"] \arrow[dd, bend right=60, "d_h"'] & & \\
D_f \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] \arrow[r, phantom, "\text{p.b.}"] & D_{g'} \arrow[d, "d_{g'}"] \arrow[dr, "\widehat{g'}"] \arrow[r, phantom, "="] & {} \\
B \arrow[r, "g"] \arrow[rr, bend right, "\widehat{=}", "h = g' \,\widehat{\circ}\, g"'] & B' \arrow[r, "g'"] & B''
\end{tikzcd}
$$

Functorial—and pullback preserving—cylindrification, with object A, inside $\mathbf{S}$, leads to:

$$
\begin{tikzcd}
A \times D_h \arrow[d, "A \times \pi_l"] \arrow[dr, "A \times \pi_r"] \arrow[ddrr, bend left, "A \times \widehat{h}"] \arrow[dd, bend right=60, "A \times d_h"'] & & \\
A \times D_g \arrow[d, "A \times d_g", "="'] \arrow[dr, "A \times \widehat{g}"] \arrow[r, phantom, "\text{p.b.}"] & A \times D_{g'} \arrow[d, "A \times d_{g'}"] \arrow[dr, "A \times \widehat{g'}"] \arrow[r, phantom, "="] & {} \\
A \times B \arrow[r, "A \times g"] \arrow[rr, bend right, "\widehat{=}", "A \times h"'] & A \times B' \arrow[r, "A \times g'"] & A \times B''
\end{tikzcd}
$$

Functoriality DIAGRAM for theory $\widehat{\mathbf{S}}$

The “global” argument for functoriality of cylindrification in $\widehat{\mathbf{S}}$ (and hence for bifunctoriality of $\times$) now reads:

Both $A \times D_h$ and $D_{(A\times g')\,\widehat{\circ}\,(A\times g)}$ are *projective Limits* of the lower-two-rows part of the $\mathbf{S}$ DIAGRAM, when coming with their respective *cones.* Therefore they admit a "comparing" *natural isomorphism,* and that's what is sufficient for functoriality of cylindrification within theory $\widehat{\mathbf{S}}$.

$\widehat{\mathbf{S}}$ inherits from $\mathbf{S}$ transposition

$$\Theta = \Theta_{A,B}(a,b) \ =_{\text{def}} \ (b,a) = (r,l) : A \times B \xrightarrow{\cong} B \times A$$

as well as diagonal

$$\Delta = \Delta_A(a) \ =_{\text{def}} \ (a,a) = (\mathrm{id},\mathrm{id}) : A \to A \times A,$$

and association

$$\begin{aligned} \mathrm{ass} = & \mathrm{ass}_{A,B,C}((a,b),c) \ =_{\text{def}} \ (a,(b,c)) = (ll,(rl,r)) : \\ & ((A \times B) \times C) \xrightarrow{\cong} (A \times (B \times C)). \end{aligned}$$

It is obvious that $\widehat{\mathbf{S}}$ inherits *naturality* of the *transformation* families ass, Θ, and Δ.

Using these natural transformations, we get (from functoriality of cylindrification) in fact *bifunctoriality* of (binary) *product* $\times$ within theory $\widehat{\mathbf{S}}$. This shows assertion (i) of the **Structure theorem.**

For **Proof** of first half of assertion (ii), namely

$$f \,\widehat{\circ}\, d_f \mathrel{\widehat{=}} \widehat{f} : A \rightharpoonup B$$

for given partial

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B,$$

consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ diagram:

$$\begin{tikzcd}
 & \mathrm{KP}[d_f] \arrow[r, "r"] \arrow[d, "l"'] \arrow[dl, bend right, "l"', start anchor=south west, end anchor=north] \arrow[dr, phantom, "\text{p.b.}"] & D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & \\
 & D_f \arrow[r, "d_f"] \arrow[rr, bend right, "\widehat{f}"'] & A \arrow[r, "f"] & B \\
D_f \arrow[ur, equal, "\mathrm{id}"'] \arrow[uur, bend left, "\Delta"] \arrow[urrr, bend right, "\widehat{\widehat{f}}=\widehat{f}"'] & & &
\end{tikzcd}$$

Partial Map Definition DIAGRAM

This diagram shows downwards inclusion

$$f \,\widehat{\circ}\, d_f = (l, \widehat{f} \circ r) \mathrel{\widehat{\subseteq}} \widehat{f} = (\mathrm{id}_{D_f}, \widehat{f}) : D_f \rightharpoonup B,$$

via $l : \mathrm{KP}[d_f] \xrightarrow{l} D_f$, with $\widehat{f}$ embedded as its graph $(\mathrm{id}_{D_f}, \widehat{f})$.

The opposite (graph) inclusion, via $\Delta : D_f \to \mathrm{KP}[\,d_f\,]$ given by reflexivity of *kernel pair* $\mathrm{KP}[\,d_f\,]$, is immediate.

For **Proof** of second $\widehat{\mathbf{S}}$ equality of assertion (ii), define opposite to $d_f : D_f \to A$ as

$$d_f^- \ =_{\mathrm{def}} \ \langle (d_f, [\ ]_{\widehat{f}}) : D_f \to A \times D_f \rangle : A \rightharpoonup D_f,$$

made *right-unique* by selecting D_f*-minimal* $\widehat{f}$ *equivalence representant*

$$[\ ]_{\widehat{f}} = [\alpha]_{\widehat{f}} \ =_{\mathrm{def}} \ \min_{D_f} \{\alpha' \leq \alpha : \widehat{f}(\alpha') \ \doteq_B \ \widehat{f}(\alpha)\} : D_f \to D_f,$$

minimal with respect to CANTOR-order on **S**-object D_f supposed pointed, by $\hat{a}_0 : \mathbb{1} \to D_f$ say.

Get in fact the commuting $\widehat{\mathbf{S}}$-DIAGRAM

$$
\begin{tikzcd}
D_f \arrow[rrd, bend left, "\widehat{f}"] & & \\
A \arrow[u, "d_f^-"] \arrow[r, phantom, "\widehat{=}"] \arrow[rr, rightharpoonup, "f"'] & & B
\end{tikzcd}
$$

Proof of (iii): For $f : A \rightharpoonup B,\ g : B \rightharpoonup C$ given in the assertion, consider—with the usual notation for defined-arguments enumerations and rules—the following DIAGRAM, showing their "total" composition $h = \langle (\mathrm{id}_A, h) : A \to A \times C \rangle : A \to C$. This DIAGRAM just enriches Composition DIAGRAM in forgoing section by the data of h and comparison $\mathbf{S}$ map $j : A \to D_{g \,\hat{\circ}\, f}$ which establishes "graph inclusion" $h \mathrel{\widehat{\subseteq}} g \,\hat{\circ}\, f : A \rightharpoonup C$.

$$
\begin{tikzcd}
D_{g \,\hat{\circ}\, f} \arrow[d, "\pi_l"] \arrow[rd, "\pi_r"] \arrow[rrdd, bend left, "\widehat{g \,\hat{\circ}\, f}"] & & \\
D_f \arrow[r, phantom, "\text{p.b.}"] \arrow[d, "d_f"] \arrow[rd, "\widehat{f}"] & D_g \arrow[r, phantom, "="] \arrow[d, "d_g"] \arrow[rd, "\widehat{g}"] & \\
A \arrow[r, rightharpoonup, "f"] & B \arrow[r, rightharpoonup, "g"] & C \\
A \arrow[u, "\mathrm{id}_A"] \arrow[rru, "h"'] \arrow[uuu, bend left, "j"] & &
\end{tikzcd}
$$

composition-total DIAGRAM for $\widehat{\mathbf{S}}$

Now define $k := \pi_l \circ j : A \to D_{g \,\hat{\circ}\, f} \to D_f$, having section property $d_f \circ k = \mathrm{id}_A : A \rightarrowtail D_f \twoheadrightarrow A$ inherited from comparison property of j :

$A \rightarrowtail D_{g \widehat{\circ} f}$. This gives the assertion, for embedded $\widetilde{f} : A \to B$, taken as $\mathbf{S}$ representant of $f : A \rightharpoonup B$, with $\widetilde{f} =_{\text{def}} \widehat{f} \circ k : A \rightarrowtail D_f \to B$.

Proof of (v): Iteration extends to partial endomaps

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times A \rangle : A \rightharpoonup A :$$

Define the domain of definition $d_{f^\S} : D_{f^\S} \to A \times \mathbb{N}$ of the (wanted) partial p. r. map $f^\S : A \times \mathbb{N} \rightharpoonup A$,

$$\begin{tikzcd}
D_{f^\S} \arrow[d, "d_{f^\S}"'] \arrow[dr, "\widehat{f^\S}"] & \\
A \arrow[r, harpoon, "f^\S"] & A
\end{tikzcd}$$

p. r. in $\mathbf{S}$ as follows: $D_{f^\S}$ is a string object

$$\begin{aligned} D_{f^\S} =_{\text{def}} \{ & \langle \hat{a}_0; \ldots; \hat{a}_{n-1} \rangle \in D_f^* : \\ & d_f(\hat{a}_0) \doteq a \wedge \bigwedge_{j=1}^{n-1} \widehat{f}(\hat{a}_j) \doteq d_f(\hat{a}_{j+1}) \} \end{aligned}$$

as free-variables diagram chase, with $a_j := d_f(\hat{a}_j)$:

$$\begin{tikzcd}
\hat{a}_0 \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \hat{a}_1 \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \cdots & \hat{a}_{n-2} \arrow[dr, mapsto, "\widehat{f}"] & \hat{a}_{n-1} \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \\
d_{f^\S}(a, n) = a & a_1 & a_2 & \cdots & a_{n-1} & a_n = \widehat{f^\S}(a, n)
\end{tikzcd}$$

Intuitively this means $(a, n) \overset{f^\S}{\mapsto} a_n \in A$, via the f-*defined arguments* $a_j \doteq_A d_f(\hat{a}_j)$:

$$A \ni a \overset{f}{\mapsto} a_1 \overset{f}{\mapsto} \ldots \overset{f}{\mapsto} a_{n-1} \overset{f}{\mapsto} a_n \in A.$$

It is clear that our zig-zag chain of applying rule $\widehat{f}$ of f and (successfully?) "searching" for a defining index—$\hat{a}_j$—for once "earlier" applying $\widehat{f}$, defines "the" right *partial map* $f^{\S} : A \times \mathbb{N} \rightharpoonup A$, and that therefore

$$f^{\S} = \langle (d_{f^{\S}}, \widehat{f^{\S}}) : D_{f^{\S}} \to (A \times \mathbb{N}) \times A \rangle : A \times \mathbb{N} \rightharpoonup A$$

constructed above, fullfills the equations for an iterated of f within theory $\widehat{\mathbf{S}}$; detailed **proof** by Peano Induction on *iteration length* n above, proof within cartesian p. r. theory $\mathbf{S}$.

This finishes the proof of the **structure theorem** for the extension $\widehat{\mathbf{S}}$, up to the **conjecture.**

3.3 Equality definability for partials

Equality definability theorem for theories $\widehat{\mathbf{S}}$ of Partial Maps: Theories $\widehat{\mathbf{S}}$ admit the following two schemes of *equality definability:*

$$(\mathrm{EqDef}_{\widehat{\subseteq}}) \quad \frac{\begin{array}{l} f,\ g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}}, \\ [\, f \ \dot{=}_B \ g\,] \mathrel{\widehat{\subseteq}} \mathrm{true}_A : A \rightharpoonup 2, \text{ i. e.} \\ \dot{=}_B \ \widehat{\circ}\, (f \times g)\, \widehat{\circ}\, \Delta_A \mathrel{\widehat{\subseteq}} \mathrm{true}_A \end{array}}{f \restriction D \ =_{\text{by def}} \ f \,\widehat{\circ}\, d \mathrel{\widehat{=}} g \restriction D \ =_{\text{by def}} \ g \,\widehat{\circ}\, d : D \to A \rightharpoonup B.}$$

Here $D = D_f \cap D_g = D_f \underset{d_f, d_g}{\times} D_g$ is the pullback—diamond in the DIAGRAM below—of the defined arguments enumerations $d_f : D_f \to A$ and $d_g : D_g \to B$ (in $\mathbf{S}$).

The original scheme, with stronger antecedent has a stronger postcedent then that expected *formally* from the general one for theories $\mathbf{T}$, namely:

$$
(\mathrm{EqDef}_{\,\widehat{=}}) \quad \frac{\begin{array}{l} f,\ g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}}, \\ [\, f \doteq_B g \,] \mathrel{\widehat{=}} \mathrm{true}_A : A \rightharpoonup \mathbb{2}, \end{array}}{\begin{array}{l} f \mathrel{\widehat{=}} \widehat{f} = \widehat{g} : A \to B \\ \ \mathrel{\widehat{=}} g : A \rightharpoonup B. \end{array}}
$$

Under the strong condition of *equality* $\widehat{=}$ with overall *truth* of $[\, f \doteq_B g \,] : A \rightharpoonup \mathbb{2}$, f and g become $\widehat{=}$ equal—as we already know—but in addition, they then necessarily admit $\widehat{=}$ *representants* within $\mathbf{S}$, namely their *rules* $\widehat{f} : D_f \to B$ resp. $\widehat{g} : D_g \to B$. Other way round: $d_f : D_f \to A$ as well as $d_g : D_g \to B$ admit sections d_f^- resp. d_g^- – within $\mathbf{S}$—and

$$
f \mathrel{\widehat{=}} \widehat{f} \circ d_f^- : A \to D_f \to B \quad \text{and} \quad g \mathrel{\widehat{=}} \widehat{g} \circ d_g^- : A \to B.
$$

Proof: For both schemes consider the following $\mathbf{S}/\widehat{\mathbf{S}}$-diagram:

$$\begin{tikzcd}
& & A \arrow[d, "\mathrm{id}"] \arrow[ddrrr, bend left, "\mathrm{true}_A"] \arrow[ddll, bend right, dotted, "j"'] & & & \\
& D_f \arrow[dr, "d_f"] \arrow[ddrrr, bend left, "\widehat{f}"] & & & & \\
D \arrow[ur, "\pi_l"] \arrow[dr, "\pi_r"'] \arrow[rr, "d"] \arrow[uurr, bend left, "i=d"] & & A \arrow[r, "\Delta"] \arrow[rr, bend right, harpoon, "(f,g)"'] & A^2 \arrow[r, harpoon, "f\times g"] & B^2 \arrow[r, "\doteq_B"] & 2 \\
& D_g \arrow[ur, "d_g"] \arrow[rrr, "\widehat{g}"'] & & & B \arrow[u, "\Delta"] \arrow[ur, "\mathrm{true}_B"'] &
\end{tikzcd}$$

partial-maps equality definability DIAGRAM

For **proof** of *inclusion* variant ($EqDef_{\widehat{\subseteq}}$) of the scheme, **S**-map $i : D \to A$ is to establish the (antecedent) inclusion

$$[\, f \ \doteq_B \ g\,] \ \widehat{\subseteq} \ \mathrm{true}_A : A \rightharpoonup 2 :$$

The diamond on the left is a pullback, to give *defined arguments enumeration* of

$$\doteq_B \ \widehat{\circ}\,(f, g) \ =_{\text{by def}} \ \doteq_B \ \widehat{\circ}\,(f \times g)\,\widehat{\circ}\,\Delta_A : A \to A^2 \rightharpoonup B^2 \to B,$$

i. e. of $(f \times g)\,\widehat{\circ}\,\Delta_A : A \rightharpoonup B^2$ as its (commuting) *diagonal* $d : D \to A$, intuitively as the *intersection* $d = d_{(f,g)} : D = D_{(f,g)} = D_f \cap D_g \to A$.

Necessarily then, $i = d : D \to A$, and commutativity of the **S**-part of the DIAGRAM shows

$$[\widehat{f} \circ \pi_l \ \doteq_B \ \widehat{g} \circ \pi_r\,] = \mathrm{true}_D : D \to B^2 \to 2,$$

and hence—by Equality Definability for theory **S** :

$$\widehat{f} \circ \pi_l = \widehat{g} \circ \pi_r : D \to B.$$

But this just means

$$f \restriction D \ =_{\text{by def}}\ f \mathbin{\widehat{\circ}} d \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} d \ =_{\text{by def}}\ g \restriction D : A \rightharpoonup B,$$

and this is what we wanted to prove in the inclusion-into-truth variant of the scheme.

In the partial-equality variant of the scheme, we have—in the DIAGRAM—additional, reverse *inclusion* $j : A \to D$, making "everything" commute, in particular $d \circ j = \mathrm{id}_A : A \to D \to A$.

From $\mathbf{S}/\widehat{\mathbf{S}}$-commutativity of the DIAGRAM, we get in particular

$$\begin{aligned}
&(f, g) \mathrel{\widehat{=}} (f, g) \mathbin{\widehat{\circ}} \mathrm{id}_A \mathrel{\widehat{=}} (f, g) \mathbin{\widehat{\circ}} d \circ j \\
&\mathrel{\widehat{=}} (f \mathbin{\widehat{\circ}} d, g \mathbin{\widehat{\circ}} d) \mathbin{\widehat{\circ}} j \mathrel{\widehat{=}} (f \mathbin{\widehat{\circ}} d_f \mathbin{\widehat{\circ}} \pi_l, g \mathbin{\widehat{\circ}} d_g \mathbin{\widehat{\circ}} \pi_r) \mathbin{\widehat{\circ}} j \\
&\mathrel{\widehat{=}} (\widehat{f} \circ \pi_l, \widehat{g} \circ \pi_r) \circ j \\
&\qquad \text{the latter by (ii) of } \textit{Structure theorem} \text{ for } \widehat{\mathbf{S}} \\
&\mathrel{\widehat{=}} (\widehat{f} \times \widehat{g}) \circ (\pi_l \circ j, \pi_r \circ j) : \\
&A \longrightarrow D_f \times D_g \xrightarrow{\widehat{f} \times \widehat{g}} B^2.
\end{aligned}$$

This shows that from $\widehat{=}$ variant of the antecedent, in fact we get representation of

$$(f, g) \ =_{\text{by def}}\ (f \times g) \circ \Delta_A : A \to A^2 \rightharpoonup B^2$$

as a p. r. map—within theory $\mathbf{S}$—and hence the same for components f and g, represented by their *rules* $\widehat{f} : D_f \to B$ and $\widehat{g} : D_g \to B$ respectively. Equality $f \mathrel{\widehat{=}} g : A \rightharpoonup B$ then follows – using this representation as $\mathbf{S}$-maps, by earlier scheme (EqDef) of equality definability for theory $\mathbf{S}$ **q. e. d.**

3.4 Partial-map extension as closure

As proved above, theory $\widehat{\mathbf{S}}$, of *partial* **S**-maps, has (cartesian) theory **S** embedded via

$$\langle f : A \to B \rangle \mapsto \langle (\mathrm{id}_A, f) : A \to A \times B \rangle : A \rightharpoonup B$$

as a *symmetric diagonal monoidal primitive recursive subtheory.*

Hence theory (theories) $\widehat{\mathbf{S}}$ turn(s) out to be—in particular—**conservative** extension(s) of theory (theories) **S**.

In analogy to diagonal monoidal *extension* $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$, the extended theory $\widehat{\mathbf{S}}$ in turn admits a—diagonal monoidal—**extension** $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}}$, into a theory of partial *partial* **S**-maps, as we show now:

objects of $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}} \sqsupset \mathbf{S}$ are to be—just as before—the objects of $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$: (bracketed, finite) *powers* of $\mathbb{N}$ and *predicative* subsets $\{A : \chi\}$ of the latter.

As *morphisms* of $\widehat{\widehat{\mathbf{S}}}$, *from* object A *to* object B, we take, in analogy of *partial* map definition over **S**, $\widehat{\mathbf{S}}$-maps of form

$$\begin{aligned}
&f = \langle \gamma f : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B,\\
&\text{where } \gamma f, \text{ as an } \widehat{\mathbf{S}} \text{ map, has (general) form}\\
&\gamma f = \langle (d_{\gamma f}, \widehat{\gamma f}) : D_{\gamma f} \to D_f \times (A \times B) \rangle : D_f \rightharpoonup A \times B,\\
&\text{with } \mathbf{S}\text{-maps (!) } d_{\gamma f} : D_{\gamma f} \to A \text{ and } \widehat{\gamma f} : D_{\gamma f} \to B.
\end{aligned}$$

$d_{\gamma f} : D_{\gamma f} \to A$ and $\widehat{\gamma f} : D_{\gamma f} \to B$ are the *components* of *graph* γf of f, which in turn defines $f : A \rightharpoondown B$ as an $\widehat{\widehat{\mathbf{S}}}$-morphism, a partial *partial* **S**-map.

For defining *composition* of such $\widehat{\widehat{\mathbf{S}}}$-morphisms, composition of, say, $f : A \rightharpoondown B$ and $g : B \rightharpoondown C$, consider the following $\mathbf{S}/\widehat{\mathbf{S}}/\widehat{\widehat{\mathbf{S}}}$-DIAGRAM which displays the $\widehat{\mathbf{S}}/\mathbf{S}$ data of f and g to be composed into an $\widehat{\widehat{\mathbf{S}}}$-morphism $g \mathbin{\widehat{\widehat{\circ}}} f : A \rightharpoondown B \rightharpoondown C$:

$$
\begin{tikzcd}
D_f \arrow[d, "\gamma f"'] & D_{\gamma f} \arrow[l, "d_{\gamma f}"'] \arrow[dl, "\widehat{\gamma f}"] & D_{g \mathbin{\widehat{\widehat{\circ}}} f} \arrow[l, "\pi_l"'] \arrow[r, "\pi_r"] \arrow[dll] \arrow[drr] \arrow[d, equal] & D_{\gamma g} \arrow[r, "d_{\gamma g}"] \arrow[dr, "\widehat{\gamma g}"'] & D_g \arrow[d, "\gamma g"] \\
A \times B \arrow[d, "l"] \arrow[drr, "r"] & & D_{\gamma f} \underset{r \circ \widehat{\gamma f},\, l \circ \widehat{\gamma g}}{\times} D_{\gamma g} \arrow[d, phantom, "\text{p.b.}"] & & B \times C \arrow[d, "r"] \arrow[dll, "l"'] \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}
$$

Composition DIAGRAM for $\widehat{\widehat{\mathbf{S}}}$

Composition $g \mathbin{\widehat{\widehat{\circ}}} f : A \rightharpoondown C$ then is defined to have as *graph* $\gamma_{g \mathbin{\widehat{\widehat{\circ}}} f}$ the map *"induced"* by the left and right *frame* morphisms of the DIAGRAM, namely:

$$
\begin{aligned}
&\gamma_{g \mathbin{\widehat{\widehat{\circ}}} f} =_{\text{def}} (l \mathbin{\widehat{\circ}} \gamma f \mathbin{\widehat{\circ}} d_{\gamma f} \circ \pi_l\,,\, r \mathbin{\widehat{\circ}} \gamma g \mathbin{\widehat{\circ}} d_{\gamma g} \circ \pi_r) : \\
&D_{g \mathbin{\widehat{\widehat{\circ}}} f} \rightharpoonup A \times C.
\end{aligned}
$$

In fact this *induced* has a representation within $\mathbf{S}$, as a *formally induced,* since

$$
\gamma f \mathbin{\widehat{\circ}} d_{\gamma f} \mathrel{\widehat{=}} \widehat{\gamma f} \quad \text{as well as} \quad \gamma g \mathbin{\widehat{\circ}} d_{\gamma g} \mathrel{\widehat{=}} \widehat{\gamma g},
$$

by assertion (ii) of the **Structure theorem** for $\widehat{\mathbf{S}}$.

[This observation already makes it plausible that the components of an $\widehat{\widehat{\mathbf{S}}}$-map can be taken—$\widehat{\widehat{\mathbf{S}}}$-*equally*—within theory $\mathbf{S}$ itself, see below]

Since composition $\,\widehat{\widehat{\circ}}\,$ of theory $\widehat{\widehat{\mathbf{S}}}$ is defined—as already composition $\,\widehat{\circ}\,$ of $\widehat{\mathbf{S}}$—by $\mathbf{S}$-pullback, it becomes associative, "again" since (finite) limits do not change—up to natural isomorphism—when limits of "subdiagrams" are added before taking the "overall" limit.

Cylindrification for theory $\widehat{\widehat{\mathbf{S}}}$ is obvious, as are then its functor properties: just cylindrify each $\mathbf{S}/\widehat{\mathbf{S}}$ diagram needed. In particular, cylindrification shows up to be functorial, with respect to composition $\widehat{\widehat{\circ}}$ for $\widehat{\widehat{\mathbf{S}}}$ introduced above. That substitutions ass, Θ, Δ—embedded into $\widehat{\widehat{\mathbf{S}}}$—have their requested *substitution* properties is obvious (embedding, as earlier, see below).

We have established so far that theory $\widehat{\widehat{\mathbf{S}}}$ defines a (symmetric) diagonal monoidal theory.

Definition of the expected embedding $\sqsubseteq\, : \widehat{\mathbf{S}} \longrightarrow \widehat{\widehat{\mathbf{S}}}$ is simple:
$\widehat{\mathbf{S}}$-maps have just $\mathbf{S}$-maps as *graphs* $\gamma f = (d_f, \widehat{f}) : D_f \to A \times B$.

[Formally we always start with such a *graph* $\gamma f : D_f \to A \times B$ "as a whole"; $d_f \;=_{\text{by def}}\; l \circ \gamma f : D_f \to A \times B \to A$, as well as $\widehat{f} \;=_{\text{by def}}\; r \circ \gamma f : D_f \to A \times B \to B$ are "only then" defined as the *components* of a graph γf given]

Mapping such graph defining $\mathbf{S}$-maps, into their $\widehat{\mathbf{S}}$-versions, defines a *symmetric diagonal monoidal embedding* $\widehat{\mathbf{S}} \sqsubseteq \widehat{\widehat{\mathbf{S}}}$, in the sense of diagonal monoidal structure of $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}}$ introduced above; in detail:

$$\begin{aligned}
&\sqsubseteq \langle f : A \rightharpoonup B \rangle \\
&=_{\text{def}} \ \langle \langle (\mathrm{id}_A, \gamma f) = (\mathrm{id}_A, (d_f, \widehat{f})) \\
&=_{\text{by def}} \ (\mathrm{id}_A, (l \circ \gamma f, r \circ \gamma f)) : A \to A \times (A \times B) \rangle : \\
&\quad A \rightharpoonup (A \times B) \rangle : A \rightharpoondown B.
\end{aligned}$$

In fact, this embedding is a Closure, as is already felt when regarding composition diagram above for theory $\widehat{\widehat{\mathbf{S}}}$. We now attempt to prove this Closure property of forming *partial partial* $\mathbf{S}$-maps.

[*Embedding* $\sqsubset : \mathbf{S} \longrightarrow \widehat{\mathbf{S}}$ was defined above by

$$\langle f : A \to B \rangle \mapsto \langle \langle (\mathrm{id}_A, f) : A \to A \times B \rangle : A \rightharpoonup B \rangle.$$

It constitutes a *functor* $\sqsubset : \mathbf{S} \longrightarrow \widehat{\mathbf{S}}$, by definition of equality $f \ \widehat{=} \ g : A \to B$ of $\widehat{\mathbf{S}}$-morphisms as *partial* $\widehat{\mathbf{S}}$-maps. As already said, this embedding preserves the diagonal monoidal structure—given on $\mathbf{S}$ as a *cartesian structure*—“into” the canonical *diagonal monoidal structure* “inherited” by $\widehat{\mathbf{S}}$ from $\mathbf{S}$. $\widehat{\mathbf{S}}$ inherits $\mathbf{S}$’s (terminal maps and) *projections* as well, but these loose their universal properties—and their character as natural transformations—within the extension]

Extension $\widehat{\widehat{\mathbf{S}}} \ \widehat{\sqsupseteq} \ \widehat{\mathbf{S}}$ “again” inherits its structure as a diagonal *monoidal theory,* “directly” from theory $\mathbf{S}$: Composition for $\widehat{\widehat{\mathbf{S}}}$ is defined by the Closure diagram below, formally an $\mathbf{S}$-DIAGRAM, the $\widehat{\mathbf{S}}$-arrows are “inserted” for orientation.

For a **proof** of the expected closure property $\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}$ consider this (commuting) $\widehat{\mathbf{S}}$ closure diagram, for a given $\widehat{\widehat{\mathbf{S}}}$ (partial *partial* $\mathbf{S}$) map

$$f = \langle \gamma f = (l \,\widehat{\circ}\, \gamma f, \ r \,\widehat{\circ}\, \gamma f) =: (d_f, \widehat{f}) : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B.$$

[Here we have applied uniqueness for the induced map which is inherited by $\widehat{\mathbf{S}}$ from $\mathbf{S}$. Starting directly with the *graph* $\gamma f : D_f \rightharpoonup A \times B$ of f, and not with its *components* d_f, $\widehat{f}$, is *necessary* (just) here, see discussion below]

$$\begin{tikzcd}
 & D_h \arrow[d, equal, "\mathrm{def}"] \arrow[ddd, bend right, "h=\gamma h"'] & \\
 & D_{\gamma f} \arrow[d, "d_{\gamma f}"] \arrow[dddl, bend right, "d_h:=\widehat{d_f}"'] \arrow[dddr, bend left, "\widehat{\widehat{f}}=:\widehat{h}"] \arrow[dd, bend left, "\widehat{\gamma f}"] & \\
 & D_f \arrow[u, bend left, "d^-_{\gamma f}"] \arrow[d, "\gamma f"] \arrow[ddl, "d_f"'] \arrow[ddr, "\widehat{f}"] & \\
 & A \times B \arrow[dl, "l"'] \arrow[dr, "r"] & \\
A \arrow[rr, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, shift right=2ex, "\widehat{\widehat{=}}"] & & B
\end{tikzcd}$$

Closure DIAGRAM for extension by partial maps

In this DIAGRAM, $\gamma f : D_f \rightharpoonup A \times B$ is the *graph* of $\widehat{\widehat{\mathbf{S}}}$-morphism $f : A \rightharpoondown B$ to be considered. The $\mathbf{S}$-maps $d_{\gamma f} : D_{\gamma f} \to D_f$ (defined-arguments enumeration) and $\widehat{\gamma f} : D_{\gamma f} \to A \times B$ (rule) are to define $\gamma f : D_f \rightharpoonup A \times B$ as a *partial* $\mathbf{S}$-map, an $\widehat{\mathbf{S}}$ morphism.

Graph $\gamma f : D_f \rightharpoonup A \times B$ has $\widehat{\mathbf{S}}$-*components*

$$\begin{aligned}
&d_f \ =_{\text{def}}\ l_{A,B} \mathbin{\widehat{\circ}} \gamma f : D_f \rightharpoonup A \times B \to A \text{ and} \\
&\widehat{f} \ =_{\text{def}}\ r_{A,B} \mathbin{\widehat{\circ}} \gamma f : D_f \rightharpoonup A \times B \to B, \\
&\text{satisfying as such—by (SP) equation for } \widehat{\mathbf{S}}\text{—} \\
&(d_f, \widehat{f}) \ =_{\text{by def}}\ (d_f \times \widehat{f}) \mathbin{\widehat{\circ}} \Delta_{D_f} \mathrel{\widehat{=}} \gamma f : D_f \to D_f \times D_f \rightharpoonup A \times B.
\end{aligned}$$

$\widehat{\mathbf{S}}$-morphism $d^-_{\gamma f} : D_f \rightharpoonup D_{\gamma f}$ is the (minimised) *opposite graph* to $\mathbf{S}$ map $d_{\gamma f}$, defined in (ii) of the **Structure theorem** for $\widehat{\mathbf{S}}$, satisfying as such $\widehat{\mathbf{S}}$ equation

$$\widehat{\gamma f} \mathbin{\widehat{\circ}} d^-_{\gamma f} \mathrel{\widehat{=}} \gamma f : D_f \rightharpoonup D_{\gamma f} \to A \times B.$$

Choice of $\widehat{\mathbf{S}}$ representant $h : A \rightharpoonup B$ for given $\widehat{\widehat{\mathbf{S}}}$ morphism $f : A \rightharpoondown B$, h to be defined by its *graph*—with components in $\mathbf{S}$—now is simple:

Take as this representant of $f : A \rightharpoondown B$, the $\widehat{\mathbf{S}}$ map $h : A \rightharpoonup B$ given by the *frame* in the DIAGRAM above:

$$\begin{aligned}
h &= \langle (d_h, \widehat{h}) : D_h \to A \times B \rangle : A \rightharpoonup B \\
&=_{\text{def}}\ \langle (\widehat{d_f}, \widehat{\widehat{f}}) : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B \\
&=_{\text{by def}}\ \langle \widehat{\gamma f} : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B.
\end{aligned}$$

As particular instance of *basic partial map* diagram in assertion (ii) of the Structure theorem for theory $\widehat{\mathbf{S}}$ we get the following commutative diagram:

$$\begin{tikzcd}
D_{\gamma f} \arrow[d, shift right, "d_{\gamma f}"'] \arrow[drr, bend left, "h=\widehat{\gamma f}"] \arrow[dr, phantom, "\widehat{=}"] & & \\
D_f \arrow[u, shift right, "d^{-}_{\gamma f}"'] \arrow[rr, harpoon, "\gamma f"] & & A \times B
\end{tikzcd}$$

Basic partial map DIAGRAM for $\widehat{\mathbf{S}}$ morphism $\gamma f : D_f \rightharpoonup A \times B$

Since $\gamma f : D_f \rightharpoonup A \times B$ is the *graph* of $f : A \rightharpoondown B$ (in $\widehat{\widehat{\mathbf{S}}}$) given, this commutative $\widehat{\mathbf{S}}$ diagram shows—by definition of equality $\widehat{\widehat{=}}$ between $\widehat{\widehat{\mathbf{S}}}$ morphisms:

$$h \mathrel{\widehat{\widehat{=}}} f : A \rightharpoondown B.$$

Embedding $\widehat{\mathbf{S}} \sqsubseteq \widehat{\widehat{\mathbf{S}}}$ is a diagonal monoidal functor, with—retractive—*Choice* $h : A \rightharpoonup B$ as *representant* for $f : A \rightharpoondown B$ in $\widehat{\widehat{\mathbf{S}}}$: Retractive Choice up to natural equivalence of functors. This is seen straightforward by definition of composition $\widehat{\widehat{\circ}}$ of $\widehat{\widehat{\mathbf{S}}}$ and of Embedding—Section $\sqsubseteq: \widehat{\mathbf{S}} \longrightarrow \widehat{\widehat{\mathbf{S}}}$.

Closure Theorem for Extension of Theory S by Partial Maps: *Closure* by *partial maps* is idempotent: Partial map closure of theory $\widehat{\mathbf{S}}$ is again a diagonal monoidal category $\widehat{\widehat{\mathbf{S}}}$ which is in fact equivalent—as such a category—to theory $\widehat{\mathbf{S}}$:

$$\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}.$$

3.5 μ-recursion without quantifiers

We **define** μ-recursion within the free-variables framework of partial p. r. maps as follows:

Given a **PR** predicate $\varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{2}$, the $\widehat{\mathbf{S}}$ morphism

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

is to have $(\mathbf{S})$ components

$$\begin{aligned}
D_{\mu\varphi} &=_{\text{def}} \{A \times \mathbb{N} : \varphi\} \subseteq A \times \mathbb{N}, \\
d_{\mu\varphi} = d_{\mu\varphi}(a,n) &=_{\text{def}} a = l \circ \subseteq : \\
&\quad \{A \times \mathbb{N} : \varphi\} \xrightarrow{\subseteq} A \times \mathbb{N} \xrightarrow{l} A, \text{ and} \\
\widehat{\mu}\varphi = \widehat{\mu}\varphi(a,n) &=_{\text{def}} \min\{m \leq n : \varphi(a,m)\} : \\
&\quad \{A \times \mathbb{N} : \varphi\} \subseteq A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}$$

Comment: This definition of $\mu\varphi : A \rightharpoonup \mathbb{N}$ is a *static* one, by enumeration $(l, \widehat{\mu}\varphi) : \{A \times \mathbb{N} : \varphi\} \to A \times \mathbb{N}$ of its *graph,* as is the case in general here for *partial* p. r. maps: We start with *given* pairs in enumeration domain $\{A \times \mathbb{N} : \varphi\}$, and get *defined arguments* a "only" as $d_{\mu\varphi}$-*enumerated* "elements" (*dependent variable*) $a = d_{\mu\varphi}(\widehat{(a,n)}) = d_{\mu\varphi}(a,n)$, $\widehat{(a,n)} = (a,n)$ "already known" to lie in $D_{\mu\varphi} = \{A \times \mathbb{N} : \varphi\}$: No need—and in general no "direct" possibility—to *decide,* for a given $a \in A$, **if** a is of form $a = d_{\mu\varphi}(a,n)$ with $(a,n) \in D_{\mu\varphi}$, i. e. if *Exists* $n \in \mathbb{N}$ such that $\varphi(a,n)$. In particular, if $D_{\mu\varphi} = \{A \times \mathbb{N} : \varphi\} = \emptyset_{A \times \mathbb{N}}$, then $d_{\mu\varphi}$ as well as $\widehat{\mu}\varphi$ are empty maps.

μ-Lemma: $\widehat{\mathbf{S}}$ admits the following (free-variables) scheme (μ) combined with $(\mu!)$—*uniqueness*—as a characterisation of the μ-operator $\langle \varphi : A \times \mathbb{N} \to \mathbb{2} \rangle \mapsto \langle \mu\varphi : A \rightharpoonup \mathbb{N} \rangle$ above:

$$(\mu)\quad \frac{\varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{2}\ \mathbf{S} - \text{map (“predicate”)},}{\begin{array}{l} \mu\varphi = \langle (d_{\mu\varphi}\,,\, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N} \\ \qquad \text{is an } \widehat{\mathbf{S}}\text{-map such that} \\ \mathbf{S} \vdash\ \varphi(d_{\mu\varphi}(\hat{a}), \widehat{\mu}\varphi(\hat{a})) = \mathrm{true}_{D_{\mu\varphi}} : D_{\mu\varphi} \to \mathbb{2}, \\ +\ \text{“argumentwise” minimality:} \\ \mathbf{S} \vdash\ [\,\varphi(d_{\mu\varphi}(\hat{a}), n) \implies \widehat{\mu}\varphi(\hat{a}) \le n\,] : D_{\mu\varphi} \times \mathbb{N} \to \mathbb{2} \end{array}}$$

as well as uniqueness—by *maximal extension*:

$$(\mu!)\quad \frac{\begin{array}{l} f = f(a) : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f} : D_f \to \mathbb{2}, \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), n) \implies \widehat{f}(\hat{a}) \le n : D_f \times \mathbb{N} \to \mathbb{2} \end{array}}{\mathbf{S} \vdash\ f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N} \ \ \text{(inclusion of graphs)}}$$

[Requiring this maximality of $\mu\varphi$ is *necessary*, since—for example—(μ) alone is fulfilled already by the *empty* partial function $\emptyset : A \rightharpoonup \mathbb{N}$]

Proof of $\mu\varphi : A \rightharpoonup \mathbb{N}$ to satisfy upper, “existence” part “(μ)” of the scheme is straigthforward by definition of $\mu\varphi$. What remains to be proved is uniqueness-by-maximal-extension scheme $(\mu!)$:

Let a partial map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

be given such that f fullfills the antecedent of scheme ($\mu!$). Then the **PR**-map

$$j = j(\hat{a}) \ =_{\mathrm{def}} \ (d_f(\hat{a}), \widehat{f}(\hat{a})) : D_f \to A \times \mathbb{N}$$

defines in fact, by the first premise on f, namely

$$\varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f}(\hat{a}) : D_f \to 2,$$

an **S**-map $j : D_f \to \{A \times \mathbb{N} : \varphi\}$ which establishes the wanted graph inclusion, namely

$$j : [\, f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}\,],$$

as shows the following $\mathbf{S}/\widehat{\mathbf{S}}$-DIAGRAM:

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[dl, "d_{\mu\varphi} = \iota"'] \arrow[d, "\varphi"] \arrow[dr, "\widehat{\mu}\varphi"] & \\
A \arrow[rr, bend right=20, harpoon, "\mu\varphi"] \arrow[rr, bend right=30, harpoon, "f"'] & 2 & \mathbb{N} \\
D_f \arrow[u, "d_f"] \arrow[uur, "j = (d_f, \widehat{f})"'] \arrow[urr, bend right, "\widehat{f}"'] & &
\end{tikzcd}$$

μ-applied-to-**S**-predicates DIAGRAM

Here, by definition of $\widehat{\mu}\varphi = \widehat{\mu}\varphi(a, n) : D_{\mu\varphi} = A \times \mathbb{N} \to \mathbb{N}$ above, we have in particular

$$\begin{aligned}
&\widehat{\mu}\varphi \circ j(\hat{a}) \ =_{\text{by def}} \ \widehat{\mu}\varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) \\
&= \min\{m \le d_f(\hat{a}) : \varphi(d_f(\hat{a}), m)\} : D_f \to A \times \mathbb{N} \to \mathbb{N}, \\
&= \widehat{f}(\hat{a}) : D_f \to \mathbb{N},
\end{aligned}$$

the latter by assumed *minimum property* of

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}.$$

Together with (trivial)

$$d_{\mu\varphi} \circ j \ =_{\text{by def}} \ l_{A,\mathbb{N}} \circ (d_f, \widehat{f}) = d_f : D_f \to A \times \mathbb{N} \to A$$

this gives in fact (remaining) *graph-inclusion* $f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}$ via $j = (d_f, \widehat{f}) : D_f \to D_{\mu\varphi} = A \times \mathbb{N}$ **q. e. d.**

Remark: Within PEANO-Arithmétique **PA**, and hence also within set theory, our $\mu\varphi : A \rightharpoonup \mathbb{N}$ equals

$$\mu\varphi = \langle (\subseteq, \widehat{\mu}\varphi) : \hat{A} \to A \times \mathbb{N} \rangle : A \supset \hat{A} \to \mathbb{N},$$

with $\hat{A} = \{\hat{a} \in A : \exists n \varphi(\hat{a}, n)\}$, and $\widehat{\mu}\varphi(\hat{a}) = \min\{m \in \mathbb{N} : \varphi(\hat{a}, m)\} : \hat{A} \to \mathbb{N}$, i. e. it is given there by the classical—partial—minimum definition. But this definition lacks *constructivity,* since $\hat{A} \subseteq A$ is in general not p. r. decidable.

What about the *Converse Direction* to μ-**Lemma** above? In fact:

Partial p. r. $\equiv$ μ-recursion, Instance of Church's Thesis: Any *partial* **S**-map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$$

is *represented*—within theory $\widehat{\mathbf{S}}$—by an " $\widehat{=}$ " equal μ-**recursive** $\widehat{\mathbf{S}}$ map, namely by

$$g = (\widehat{f} \circ \mathrm{count}_{D_f}) \mathbin{\widehat{\circ}} \mu\varphi_f : A \rightharpoonup \mathbb{N} \to D_f \to B,$$

$\varphi_f = \varphi_f(a, n) : A \times \mathbb{N} \to \mathbb{2}$ suitable, namely

$$\varphi_f = \varphi_f(a, n) \ =_{\text{def}} \ [\, a \ \dot{=}_A \ d_f \circ \text{count}_{D_f}(n)\,] : A \times \mathbb{N} \to \mathbb{2} \text{ (PR)},$$

$\text{count}_{D_f} : \mathbb{N} \to D_f$ being a Cantor type (PR) *count* of D_f.

Joker Remark:

$$\text{count}_{D_f} = \text{count}_{D_f}(n) : \mathbb{N} \to D_f = \{\mathbb{X} : D_f : \mathbb{X} \to \mathbb{2}\}$$

is easily constructed if D_f comes with a *point,* $\hat{a}_0 : \mathbb{1} \to D_f$ say. If not—or if you cannot name such point—, just add one "as a joker", namely injection $\iota : \mathbb{1} \to \mathbb{1} + D_f$ into the sum, replace D_f by $\mathbb{1} + D_f$, A by $\mathbb{1} + A$, B by $\mathbb{1} + B$, d_f by $\mathbb{1} + d_f : \mathbb{1} + D_f \to \mathbb{1} + A$, $\widehat{f}$ by $\mathbb{1} + \widehat{f} : \mathbb{1} + D_f \to \mathbb{1} + B$, and keep track of the joker.

So D_f is "now" pointed, and admits—because of this—a retractive count $\text{count}_{D_f} : \mathbb{N} \to D_f$, by linear (well) order on D_f, inherited from that of $\mathbb{X}$, and anchored at D_f's *point,* "defined element" $\hat{a}_0 : \mathbb{1} \to D_f \sqsubset \mathbb{X}$.

Proof of Partials to be μ-recursive maps:

Consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ diagram:

$$\begin{tikzcd}
 & D_f \arrow[d, "d_f"] \arrow[drr, "\widehat{f}"] \arrow[ddl, bend right, "i"'] \arrow[dr, phantom, "\widehat{=}"] & & \\
 & A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "\mu\varphi_f"] \arrow[d, phantom, "\widehat{=}"] \arrow[l, phantom, "="] & \arrow[dr, phantom, "\underset{(\text{def})}{\widehat{=}}"] & B \\
D_{\mu\varphi_f} \arrow[ur, "d_{\mu\varphi_f}"] \arrow[rr, "\widehat{\mu}\varphi_f"'] \arrow[uur, bend left, "j"] & & \mathbb{N} \arrow[r, "\text{count}"'] & D_f \arrow[u, "\widehat{f}"']
\end{tikzcd}$$

Partial p. r. Map $\equiv$ μ-recursion diagram

All objects and (partial) maps in this DIAGRAM have been defined above, with the exception of (PR) *comparison maps* $i : D_f \to D_{\mu_f}$, and j in the other direction.

We define these two maps "suitably", by

$$\begin{aligned}
&D_{\mu\varphi_f} \;=_{\text{by def}}\; \{A \times \mathbb{N} : \varphi_f\} \;=_{\text{by def}}\; \{(a,n) : d_f \circ \mathrm{count}_{D_f}(n) \doteq_A a\}, \\
&i = i(\hat{a}) \;=_{\text{def}}\; (d_f(\hat{a}), \min\{m \le n : d_f(\mathrm{count}_{D_f}) \doteq_A d_f(\hat{a})\} : D_f \to D_{\mu\varphi_f}, \\
&\text{and } j = j(a,n) \;=_{\text{def}}\; \mathrm{count}_{D_f}(\min\{m \le n : d_f(\mathrm{count}(m)) \doteq a\}) : \\
&A \times \mathbb{N} \supseteq D_{\mu\varphi_f} \to D_f.
\end{aligned}$$

By definition of $\varphi_f : A \times \mathbb{N} \to \mathbb{2}$, and then—general for such a predicate, see above—of

$$\mu\varphi_f = \langle (d_{\mu\varphi_f}, \widehat{\mu}\varphi_f) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N},$$

and—eventually—(alledged) *representant*

$$g \;=_{\text{def}}\; \widehat{f} \circ \mathrm{count}_{D_f} \,\widehat{\circ}\, \mu\varphi_f : A \rightharpoonup \mathbb{N} \to D_f \to B,$$

of f, this $\widehat{\mathbf{S}}$-DIAGRAM commutes; μ-recursive *representant* involves just (two) **S**-maps, namely—PR retraction count $= \mathrm{count}_{D_f} : \mathbb{N} \twoheadrightarrow D_f$, and *rule* $\widehat{f} : D_f \to B$ (given)—, as well as one genuinely μ-recursive map $\mu\varphi_f : A \rightharpoonup \mathbb{N}$: μ-recursion applied to **S**-predicate $\varphi_f : A \times \mathbb{N} \to \mathbb{2}$. Commutativity of this $\widehat{\mathbf{S}}$-DIAGRAM shows

$$i : [\, f \,\widehat{\subseteq}\, g : A \rightharpoonup B \,],\; j : [\, g \,\widehat{\subseteq}\, f : A \rightharpoonup B \,], \text{ and hence } f \,\widehat{=}\, g : A \rightharpoonup B :$$

An arbitrary *partial* p. r. map $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$ admits, within $\widehat{\mathbf{S}}$, a representation $g : A \rightharpoonup B$, obtained via suitable **S**-map(s) and one μ-recursive one, $\mu\varphi_f : A \rightharpoonup \mathbb{N}$, defined in turn "over" an **S**-predicate,namely $\varphi_f : A \times \mathbb{N} \to \mathbb{2}$ above **q. e. d.**

Corollary: define theory $\mu\mathbf{S}$, over $\mathbf{S}$ and within $\widehat{\mathbf{S}}$, by Closure of $\mathbf{S}$ under the μ-operator—applied to $\mathbf{S}$-predicates—*merged* with Monoidal-theory Closure. Then this *subtheory* $\mu\mathbf{S}$ is in fact isomorphic to theory $\widehat{\mathbf{S}}$, as a Diagonal Monoidal p. r. theory:

$$\mathbf{S} \sqsubset \mu\mathbf{S} \cong \widehat{\mathbf{S}}.$$

Both theories have cartesian p. r. theory $\mathbf{S}$ embedded as diagonal monoidal subcategory, and the embedding is compatible with the isomorphism $\mu\mathbf{S} \cong \widehat{\mathbf{S}}$.

By foregoing theorem, and the Closure theorem of section above for theorie(s) $\widehat{\mathbf{S}}$—strengthening theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$—we are sure that we can extend the μ-operator—already within theory $\widehat{\mathbf{S}}$—to *partial* predicates

$$\varphi = \langle (d_\varphi, \widehat{\varphi}) : D_\varphi \to (A \times \mathbb{N}) \times \mathbb{2} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2},$$

and that the $\widehat{\mathbf{S}}$-morphism

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

—if suitably defined along Partial-Map Closure Terminology—inherits suitably generalised charateristic properties from the μ-operator applied to p. r. predicate φ above, i. e. to $\varphi : A \times \mathbb{N} \to \mathbb{2}$ in $\mathbf{S}$.

But may be a *direct* definition and characteristic scheme for the μ-operator on (partially defined) predicates

$$\varphi = \langle \gamma\varphi = (l \circ \gamma\varphi, r \circ \gamma\varphi) =: (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_\varphi \to A \times \mathbb{N} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2}$$

is less complicate—and more instructive:

Definition: Given an $\widehat{\mathbf{S}}$-predicate $\varphi : A \times \mathbb{N} \rightharpoonup 2$, we take as *Enumeration domain* $D_{\mu\varphi}$ for (the graph of) $\mu\varphi : A \rightharpoonup \mathbb{N}$ to be constructed,

$$D_{\mu\varphi} \ =_{\text{def}} \ \{D_\varphi : \widehat{\varphi}\} \ =_{\text{by def}} \ \{\alpha \in D_\varphi : \widehat{\varphi}(\alpha)\},$$

and as *components* of $\mu\varphi$:

$$d_{\mu\varphi} \ =_{\text{def}} \ l \circ d_\varphi \circ \subseteq : \{D_\varphi : \widehat{\varphi}\} \to D_\varphi \to A \times \mathbb{N} \to A, \text{ and}$$
$$\widehat{\mu\varphi} = \widehat{\mu\varphi}(\alpha) \ =_{\text{def}} \ \min\{m \leq r \circ d_\varphi \circ \subseteq \ (\alpha)\} :$$
$$\{D_\varphi : \widehat{\varphi}\} \xrightarrow{\subseteq} D_\varphi \xrightarrow{d_\varphi} A \times \mathbb{N} \to \mathbb{N} \to \mathbb{N},$$

cf. "μ-applied-to-partial-predicates DIAGRAM in Proof of next theorem.

[It is obvious that restriction of the above *extended* μ-operator to $\mathbf{S}$-predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \to 2$ coincides with the one given at begin of section for this case: In that p. r. case for φ, $D_\varphi = A \times \mathbb{N}$ etc.]

We now generalise the characteristic μ-schemes for the PR, $\mathbf{S}$-predicates $\varphi : A \times \mathbb{N} \to 2$, to the case of *partial* ones $\varphi : A \times \mathbb{N} \rightharpoonup 2$, and **prove** that $\mu\varphi : A \rightharpoonup A$ defined above for that general case of φ, *fullfills* the schemes, in particular the scheme of *uniqueness by maximality,* within theory $\widehat{\mathbf{S}}$ of *partial* $\mathbf{S}$-maps.

Characterisation Theorem for the μ-operator applied to *partial* predicates $\varphi : A \times \mathbb{N} \rightharpoonup 2$:

Theory $\widehat{\mathbf{S}}$ admits characterisation of its μ-operator introduced above, by the following (general) μ-**scheme:**

$$(\mu)_{\widehat{\mathbf{S}}}\quad \frac{\varphi = \langle (d_\varphi, \widehat{\varphi}) \rangle : A \times \mathbb{N} \rightharpoonup 2 \ \ \widehat{\mathbf{S}}\text{-}\mathit{predicate},}{\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}}$$

an $\widehat{\mathbf{S}}$-morphism, i. e. a *partial* **S**-map, such that

$$\mathrm{true}_A \mathrel{\widehat{\supseteq}} \varphi \mathbin{\widehat{\circ}} (\mathrm{id}_A \times \mu\varphi) \mathbin{\widehat{\circ}} \Delta_A : A \to A \times A \rightharpoonup (A \times \mathbb{N}) \rightharpoonup 2,$$

in diagonal monoidal free-variables "Calculus":

$$\mathrm{true}_A \mathrel{\widehat{\supseteq}} \varphi \mathbin{\widehat{\circ}} \ (a, \mu\varphi \mathbin{\widehat{\circ}} a) : A \rightharpoonup 2.$$

+ *minimality* (FV):

$$\mathrm{true}_{A\times\mathbb{N}} \mathrel{\widehat{\supset}} [\, \varphi \mathbin{\widehat{\circ}} (a, n) \implies \mu\varphi \mathbin{\widehat{\circ}} a \le n \,] : A \times \mathbb{N} \rightharpoonup 2,$$

with *free variables* $a \in A$ and $n \in \mathbb{N}$ interpreted as *identities:*

$$\begin{aligned}\mathrm{true}_{A\times\mathbb{N}} \mathrel{\widehat{\supset}} \ \implies \ \widehat{\circ}\,(\varphi,\ \le \widehat{\circ}\,(\mu\varphi, r)) :\\ A \times \mathbb{N} \rightharpoonup (A \times \mathbb{N}) \times \mathbb{N}^2 \rightharpoonup 2^2 \to 2,\end{aligned}$$

where—as usual—"induced" partial maps are just taken as **abbreviations** for the "official" versions defined via *diagonals.*

+ *uniqueness* of $\mu\varphi : A \rightharpoonup \mathbb{N}$ by maximal-graph property:

$$(\mu!)_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} f : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ (given) such that} \\ f : A \rightharpoonup \mathbb{N} \text{ in place of } \varphi \text{ satisfies} \\ \text{all of the above graph inclusions,} \\ \text{(in particular } \mathit{minimality}) \end{array}}{\mathbf{S} \vdash \ f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}.}$$

Proof: That our $\mu\varphi : A \rightharpoonup \mathbb{N}$ defined above for partial predicate $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$ satisfies the *basic* schemes (μ) and *minimality* above, is seen straightforward.

But what about *graph-maximality* with respect to other partial maps

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N},$$

equally satisfying these two conditions?

We generalise our earlier proof of this graph-maximality, with respect to **S**-predicates φ, to the case of a *partial* one,

$$\varphi = \langle (d_\varphi, \widehat{\mu}\varphi) : D_\varphi \to (A \times \mathbb{N}) \times \mathbb{2} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2}, \text{ in } \widehat{\mathbf{S}},$$

by enriching the earlier "μ-applied-to-**S** DIAGRAM" with the new data for the φ-*partial* case:

$$
\begin{tikzcd}
& & D_{\mu\varphi} = \{D_\varphi : \widehat{\varphi}\} \arrow[d, "\subseteq"] \arrow[dddll, "d_{\mu\varphi}"'] \arrow[dddrr, "\widehat{\mu\varphi}"] & & \\
& & D_\varphi \arrow[d, shift left, "d_\varphi"] \arrow[dd, bend left, "\widehat{\varphi}"] & & \\
& & A \times \mathbb{N} \arrow[u, shift left, harpoon, "d_\varphi^-"] \arrow[d, "\varphi"] \arrow[dll, "l"'] \arrow[drr, "r"] & & \\
A \arrow[rrrr, harpoon, "\mu\varphi"] \arrow[rrrr, harpoon, bend right, "f"'] & & 2 & & \mathbb{N} \\
D_f \arrow[u, "d_f"] \arrow[uuurr, "j"] \arrow[uurr, "(d_f{,}\widehat{f})"'] \arrow[urrrr, bend right, "\widehat{f}"'] \arrow[uuuurr, harpoon, bend left, "\bar{\jmath}"] & & & &
\end{tikzcd}
$$

μ-applied-to-partial-predicates DIAGRAM

The lower part of the DIAGRAM is to display the data (components) of such a "candidate" $f : A \rightharpoonup \mathbb{N}$. What we need for the asserted inclusion $f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}$ is a *commutative fill-in* into the diagram, from domain D_f to domain $D_{\mu\varphi} = \{D_\varphi : \widehat{\varphi}\}$. By the closure theorem of foregoing section, a *partial map* commutative fill-in $\bar{\jmath} : D_f \rightharpoonup \{D_\varphi : \widehat{\varphi}\}$ is enough. For "constructing" this, take as a—non-trivial, new—building block, partial map $d_\varphi^- : A \times \mathbb{N} \rightharpoonup D_\varphi$, opposite (as graph) to $d_\varphi : D_\varphi \to A \times \mathbb{N}$ given, cf.**Structure theorem** for theory $\widehat{\mathbf{S}}$, assertion (ii): Basic Partial Map DIAGRAM for $f : A \rightharpoonup B$, here for $\varphi : A \times \mathbb{N} \rightharpoonup 2$:

This opposite d_φ^- has, by that theorem, the *typical property*

$$\widehat{\varphi} \mathbin{\widehat{\circ}} d_\varphi^- \mathrel{\widehat{=}} \varphi : A \times \mathbb{N} \rightharpoonup D_\varphi \to 2.$$

So we get a precursor for realising *graph-inclusion,* namely

$$j \ =_{\mathrm{def}} \ d_\varphi^- \,\widehat{\circ}\, (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup D_\varphi.$$

Because of

$$\begin{aligned}
&\widehat{\varphi} \,\widehat{\circ}\, j : D_f \rightharpoonup D_\varphi \\
&\quad =_{\text{by def}} \ \widehat{\varphi} \,\widehat{\circ}\, d_\varphi^- \circ (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup D_\varphi \to 2 \\
&\quad \widehat{=} \ \varphi \,\widehat{\circ}\, (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup 2 \ \ \text{(see just above)} \\
&\quad \widehat{=} \ \mathrm{true}_{D_f} : D_f \rightharpoonup 2 \ \ \text{(by comparison condition on } f : A \rightharpoonup \mathbb{N}) \\
&\quad \widehat{\subseteq} \ \mathrm{true}_A : A \to 2 \text{ via } d_f \to A,
\end{aligned}$$

we get in particular

$$\widehat{\varphi} \,\widehat{\circ}\, j \ \widehat{=} \ \mathrm{true}_{D_\varphi} : D_f \to D_\varphi \to 2.$$

Since the *universal* equaliser property of predicative extension

$$\{D_\varphi : \widehat{\varphi}\} \xrightarrow{\subseteq} D_\varphi,$$

equaliser of $\widehat{\varphi}$, $\mathrm{true}_{D_\varphi} : D_\varphi \to 2$, is preserved by embedding $\mathbf{S} \sqsubset \widehat{\mathbf{S}}$, the above eventually "generates" the lacking $\widehat{\mathbf{S}}$ morphism

$$\bar{\jmath} : D_f \rightharpoonup D_{\mu\varphi} = \{D_\varphi : \widehat{\varphi}\}$$

which establishes, by the shown $\widehat{=}$ commutativities, (graph) *inclusion*

$$f \ \widehat{\widehat{\subseteq}} \ \mu\varphi : A \rightharpoonup \mathbb{N},$$

and hence by closure property of embedding $\sqsubseteq : \widehat{\mathbf{S}} \xrightarrow{\cong} \widehat{\widehat{\mathbf{S}}}$ the characteristic properties of the μ-operator, here in case of application to *partial* predicates $\varphi : A \times \mathbb{N} \rightharpoonup 2$ **q. e. d.**

[It is obvious that Definition and general scheme $(\mu)_{\widehat{\mathbf{S}}}$ above, restrict to earlier (definition resp.) scheme $(\mu) = (\mu)_{\mathbf{S}}$ for PR, **S**-predicates $\varphi : A \times \mathbb{N} \to \mathbb{2}$]

Our **conclusion** so far is:

- We can *eliminate formal existential quantification* —as well as (individual, formal) *variables*— from the theory of μ-recursion, by interpreting the μ-operator from theories **S** of primitive recursion into their respective extensions $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$ by *partial* p. r. maps.

- The μ-operator canonically extends to all *partial* predicates $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$, and associates to them just partial maps $\mu\varphi : A \rightharpoonup \mathbb{N}$, within $\widehat{\mathbf{S}}$ itself. So, "once again", we see, that theories $\widehat{\mathbf{S}}$ of *partial p. r. maps* are *closed,* this time under the μ-operator, "in parallel" to *Closure* of $\widehat{\mathbf{S}}$ under forming *partial* maps: *partial partial p. r. maps* "are" *partial* p. r. maps.

- We have the following chain of isomorphisms of categorical theories considered so far:

  $$\mathbf{S} \sqsubset \mu\mathbf{S} \cong \mu\mu\mathbf{S} \cong \widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}} \sqsupset \mathbf{S},$$

  the embeddings being *diagonal-monoidal p. r. compatible* with the isomorphisms.

  [A *partial* p. r. map $f : A \rightharpoonup B$ which is, "by hazard", a total map—discussion of overall *termination* = *total definedness* in section 2—is in general *not* itself PR: only its graph $(d_f, f) : D_f \to A \times B$ is PR. Ackermann type maps, in particular *evaluation* of all PR-map-codes, are *counter examples.*]

- Conversely, the μ-operator, already when applied only to **S**-predicates: p. r. predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, *generates* all $\widehat{\mathbf{S}}$-morphisms—*partial* **S**-maps—out of **S**, via necessarily formally *partial* composition with suitable **S**-maps.

- As important special cases of basic p. r. theories **S** we have at the moment theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$, Universe p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$, as well as the p. r. *trace* $\mathbf{PA} \restriction \mathrm{PR}$ of **PA** : All p. r. maps with all those equations in between, which are derivable by **PA** : Our theories, notions, and results have a structure-preserving interpretation "into" (within) Peano-arithmetic **PA**, a fortiori into classical **set** theory.

3.6 Content driven loops

By a *content driven* loop we mean an *iteration* of a given *step endo map,* whose number of performed steps is not known at *entry time* into the *loop*—as is the case for a PR iteration $f^{\S}(a, n) : A \times \mathbb{N} \to A$ with *iteration number* $n \in \mathbb{N}$—, but whose (re) entry into a "new" endo step $f : A \to A$ depends on *content* $a \in A$ reached so far:

This *(re) entry* or *exit* from the loop is now *controlled* by a *(control) predicate* $\chi = \chi(a) : A \to \mathbb{2}$.

First example: a <u>while</u> loop $\mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$, for given p. r. *control* predicate $\chi = \chi(a) : A \to \mathbb{2}$, and *(looping) step* endo $f : A \to A$, both in **S**, both **S**-maps for the time being, **S** as always in our present context an extension of **PRa**, admitting the scheme of (predicate) *abstraction.* Examples for the moment: $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ itself, Universe theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ as well as $\mathbf{PA} \restriction \mathrm{PR}$, restriction of **PA** to its p. r. terms, with inheritance of all **PA**-equations for this term-restriction.

Classically, *with* variables, such $\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,]$ would be "defined"—in *pseudocode*—by

$$\begin{aligned}\mathrm{wh}(a) := [\,a' &:= a;\\ &\underline{\text{while}}\ \chi(a')\ \underline{\text{do}}\ a' := f(a')\ \underline{\text{od}};\\ &\mathrm{wh}(a) := a'\,].\end{aligned}$$

The formal version of this—within a *classical,* element based setting—, is the following partial-(PEANO)-map characterisation:

$$\mathrm{wh}(a) = \mathrm{wh}\,[\,\chi : f\,]\,(a) = \begin{cases} a \text{ if } \neg\,\chi(a) \\ \mathrm{wh}(f(a)) \text{ if } \chi(a) \end{cases} \quad : A \rightharpoonup A.$$

But can this *dynamical, bottom up* "definition" be converted into a p. r. *enumeration* of a suitable *graph* "of all *argument-value pairs*" in terms of an $\widehat{\mathbf{S}}$-morphism

$$\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] = \langle (d_{\mathrm{wh}}, \widehat{\mathrm{wh}}) : D_{\mathrm{wh}} \to A \times A \rangle : A \rightharpoonup A?$$

In fact, we can give such *suitable,* static Definition of $\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$—within $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—as follows:

$$\begin{aligned}&\mathrm{wh} \ =_{\mathrm{def}}\ f^{\S}\,\widehat{\circ}\,(\mathrm{id}_A, \mu\,\varphi_{[\,\chi\,|\,f\,]})\\ &\quad =_{\text{by def}}\ f^{\S}\,\widehat{\circ}\,(A \times \mu\,\varphi_{[\,\chi:f\,]})\,\widehat{\circ}\,\Delta_A :\\ &A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \ \text{where}\\ &\varphi = \varphi_{[\,\chi:f\,]}(a,n) \ =_{\mathrm{def}}\ \neg\,\chi\,f^{\S}(a,n) : A \times \mathbb{N} \to A \to 2 \to 2.\end{aligned}$$

Within a quantified arithmetical theory like **PA**, this $\widehat{\mathbf{S}}$-definition of $\mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$ fullfills the classical characterisation quoted above, as is readily shown by Peano-Induction "on" $n := \mu\,\varphi_{[\,\chi:f\,]}\,(a) : A \rightharpoonup \mathbb{N}$, at least within **PA** and its extensions.

[Classically, *partial definedness* of this—*dependent*—induction parameter n causes no problem: use a *case distinction* on definedness of $\mu\,\varphi_{\chi,f}(a)$ "$\in$" $\mathbb{N}$. Even in our quantifier-free context such *dependent induction* on a *partial* dependent induction parameter will be available, see below]

In this generalised sense, we have—within theories $\widehat{\mathbf{S}} \ \sqsupset\ \mathbf{S}$—all <u>while</u> loops, for the time being at least those with *control* $\chi : A \to \mathbb{2}$ and *step* endo $f : A \to A$ within **S**.

It is obvious that such $\mathrm{wh}\,[\,\chi : f\,] : A \times A$ is in general "only" *partial*—as is trivially exemplified by integer division by *divisor* 0, which would be endlessly subtracted from the dividend, although in this case *control* and *step* are both PR.

By the classical characterisation of these <u>while</u> loops above, we are motivated for its generalisation to the $\mathbf{S}/\widehat{\mathbf{S}}$ case:

Characterisation Theorem for <u>while</u> loops *over* **S**, within theory $\widehat{\mathbf{S}}$: For $\chi : A \to \mathbb{2}$ (*control*) and $f : A \to A$ (*step*), both—for the time being—**S**-maps, <u>while</u> loop $\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$ (as defined above), is characterised by the following *implications* within $\widehat{\mathbf{S}}$:

$$\widehat{\mathbf{S}} \vdash\ \neg\,\chi \circ a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq a : A \rightharpoonup \mathbb{2},\ \text{and}$$

$$\widehat{\mathbf{S}} \vdash\ \chi \circ a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq \mathrm{wh}\,\widehat{\circ}\,f \circ a.$$

where use of "sort of" free variable 'a' is to help intuition, *formally* a is just another name for $\mathrm{id}_A : A \to A$, more precisely:

Rudiments of a free-variables Calculus "over" a diagonal symmetric monoidal (PR) theory:

– An identity $\mathrm{id} : A \to A$—*not* a projection in general—can be seen as a *free variable,* "ranging over" those "arguments" of A, for which the *partial map* in question—to be "applied" to that "argument"—is "thought to be defined".

– For $a \in A,\ b \in B$ free, (a, b) is—again, as in the cartesian case—interpreted as the identity $(a \times b) = id_{A\times B} : (A \times B) \to (A \times B)$, here for given *partial* maps $f : A \rightharpoonup A',\ g : B \rightharpoonup B'$ as in

$$\begin{aligned}
&(f \times g) \,\widehat{\circ}\, (a, b)\\
&\quad =_{\text{by def}} \ (f \times g) \,\widehat{\circ}\, (\mathrm{id}_A \times \mathrm{id}_B)\\
&\quad \widehat{=} \ ((f \,\widehat{\circ}\, a) \times (g \,\widehat{\circ}\, b)) : A \times B \rightharpoonup A' \times B'.
\end{aligned}$$

– By *transposition* Θ, such (identity) free-variables a and b may be interchanged—Θ may in that case become *implicit.*

– A *diagonal* Δ may *double* such an identity free-variable, and become implicit in turn.

Problematic is in the *partial maps* case, introduction of terminal maps $\Pi : A \to \mathbb{1}$, and even more of projections, since in general, for $f : C \rightharpoonup A,\ g : C \rightharpoonup B$, say within $\widehat{\mathbf{S}}$, only

$$f \,\widehat{\supseteq}\, l \,\widehat{\circ}\, (f, g) \ =_{\text{by def}} \ (f \times g) \,\widehat{\circ}\, \Delta_C : C \to C^2 \to A \times B \to A,$$

analogeously for g : in general we have genuine *graph inclusions,* since

$$D_{(f,g)} = D_{(f\times g)\,\widehat{\circ}\,\Delta_C} \ =_{\text{by def}} \ D_f \cap D_g$$

is in general a non-trivial pullback, not isomorphic to C as in the cartesian case.

So—for the time being—if you want to "use" (general) projections as free variables, you must take care of the *lack of naturality* of the projection family in a general diagonal monoidal setting, or—formally—try to replace cartesian products $A \xleftarrow{l} A \times B \xrightarrow{r} B$ by *half-trivial* pullbacks

$$A \xleftarrow{l} A \underset{\mathbb{1}}{\times} B \xrightarrow{r} B\ ,$$

inherited from theory **S**. *Universality* of such a pullback *over* object $\mathbb{1}$—with its arrows $\Pi : A \to \mathbb{1}$ and $\Pi : B \to \mathbb{1}$ given—"admits" only those $\widehat{\mathbf{S}}$ morphism pairs into its factors, which have *equal domains of defined arguments.*

Using these tentative rules for a free-variables Calculus interpretation, the statement of our **Characterisation Theorem** above for <u>while</u> loops, takes the following purely morphism theoretic form in theory $\widehat{\mathbf{S}}$:

$$\begin{aligned} \mathrm{true}_A \mathrel{\widehat{=}}\ \ &\implies\ \widehat{\circ}\,(\neg\,\chi,\ \dot{=}_A\ \ \widehat{\circ}\,(\mathrm{wh}, \mathrm{id}_A)) : A \rightharpoonup A,\ \ \textit{and} \\ \mathrm{true}_A \mathrel{\widehat{=}}\ \ &\implies\ \widehat{\circ}\,(\chi,\ \dot{=}_A\ \ \widehat{\circ}\,(\mathrm{wh}, \mathrm{wh}\,\widehat{\circ}\,f)) : A \rightharpoonup A. \end{aligned}$$

We begin with the **Proof** of wh to be unique with regard to fullfill the "<u>while</u>-characterisation", by Peano-Induction on $m := \mu[\,\chi : f\,] : A \rightharpoonup \mathbb{N}$, more precisely: by Peano-Induction on *dependent induction-parameter*

$$m \doteq \widehat{\mu}[\,\neg\,\chi : f\,](\hat{a}) : D_\mu \to \mathbb{N}.$$

This needs a **Dependent Induction Parameter Peano Induc-**

tion scheme for theory $\mathbf{S}$ which reads as follows:

$$\begin{array}{ll}
 & c = c\,(a) : A \to \mathbb{N}, \text{ (complexity)} \\
 & \chi = \chi(a) : A \to \mathbb{2}, \text{ (predicate under consideration),} \\
 & \qquad \text{both in } \mathbf{S}, \\
 & \mathbf{S} \vdash\ c\,(a) \doteq 0 \implies \chi(a) : A \to \mathbb{2}, \text{ (anchor)} \\
 & \mathbf{S} \vdash\ [\,c\,(a) \doteq n \implies \chi(a)\,] \\
 & \qquad \implies [\,c\,(a') \doteq s\,n \implies \chi(a')\,] : A^2 \times \mathbb{N} \to \mathbb{2} \text{ (step)} \\
(P5^*) & \overline{\hspace{25em}} \\
 & \mathbf{S} \vdash\ \chi(a) : A \to \mathbb{2}.
\end{array}$$

Proof by Peano-Induction—available in p. r. theories via Freyd's uniqueness: apply this Peano Induction to predicate

$$\varphi = \varphi(a, n) := [\,c\,(a) \doteq n \implies \chi(a)\,] : A \times \mathbb{N} \to \mathbb{2},$$

and get overall truth of this $\varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, and hence trivially $\mathbf{S} \vdash\ \chi : A \to \mathbb{2}$, by substitution of $c\,(a)$ into n.

Comment: This scheme $(P5^*)$ holds true in all *diagonal monoidal* PR theories—for example in $\widehat{\mathbf{S}}\ \sqsupset\ \mathbf{S}$—since Peano-Induction is a consequence already of Freyd's uniqueness (FR!), which is available by axiom—in case of extensions $\widehat{\mathbf{S}}$ of p. r. *cartesian* theories $\mathbf{S}$ as **theorem.**

For **proof** of uniqness of wh in this general, diagonal monoidal case it would certainly be helpful if we could build on a suitable generalisation of the *cartesian* **free-variables Calculus,** generalisation to a—necessarily restricted—form, applicable to the *diagonal (symmetric) monoidal* case of p. r. theories. For "rudiments" of such an FV-Calculus see above.

We extend the μ-based definition of wh in $\widehat{\mathbf{S}}$ formally into the case of *partial* $\chi : A \rightharpoonup 2$ and $f : A \rightharpoonup A$ by use of the μ-operator, as follows:

$$\begin{aligned}
&\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] \;=_{\mathrm{def}}\; f^{\S}\,\widehat{\circ}\,(\mathrm{id}_A, \mu\,\varphi_{[\chi,f]}) \\
&\quad =_{\text{by def}}\; f^{\S}\,\widehat{\circ}\,(A \times \mu\,\varphi_{[\chi,f]})\,\widehat{\circ}\,\Delta : \\
&A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \text{ where} \\
&\varphi = \varphi_{[\chi,f]} \;=_{\mathrm{def}}\; \neg\,\widehat{\circ}\,\chi\,\widehat{\circ}\,f^{\S} : A \times \mathbb{N} \rightharpoonup A \rightharpoonup 2 \to 2,
\end{aligned}$$

this time—last line—iteration $f^{\S} : A \times \mathbb{N} \rightharpoonup A$ of *partial* $f : A \rightharpoonup A$ defined—and characterised—via the "zig/zag" way in the proof of structure theorem above.

Characterisation Theorem for <u>while</u> Loops:
For (partial) *control*

$$\begin{aligned}
&\chi \mathrel{\widehat{=}} \chi\,\widehat{\circ}\,a : A \rightharpoonup 2, \text{ and } \textit{(endo) step} \\
&f \mathrel{\widehat{=}} f\,\widehat{\circ}\,a : A \rightharpoonup A,
\end{aligned}$$

("Content driven") <u>while</u> loop

$$\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$$

is characterised—within theory $\widehat{\mathbf{S}}$ of partial $\mathbf{S}$ maps—by

$$\begin{aligned}
&\mathrm{true}_A \mathrel{\widehat{=}} [\,\neg\,\chi\,\widehat{\circ}\,a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq a\,] : A \rightharpoonup 2 \textit{ and} \\
&\mathrm{true}_A \mathrel{\widehat{=}} [\,\chi\,\widehat{\circ}\,a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq \mathrm{wh}\,\widehat{\circ}\,f\,\widehat{\circ}\,a\,] : A \rightharpoonup 2.
\end{aligned}$$

Proof: We use the following abbreviations:

$$\begin{aligned}&\mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a := \mathrm{wh}\,[\,\chi, f\,] \mathbin{\widehat{\circ}} a : A \rightharpoonup A, \text{ and}\\ &\mu \mathrel{\widehat{=}} \mu \mathbin{\widehat{\circ}} a := \mu\{A \times \mathbb{N} : \neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S}\}\\ &\ \mathrel{\widehat{=}} \mu\{(a, n) \in A \times \mathbb{N} : \neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S} \mathbin{\widehat{\circ}} (a, n)\}.\end{aligned}$$

That $\mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a : A \rightharpoonup A$ fullfills the implications of (alledged) characterisation is obvious.

For showing uniqueness of such partial map wh, assume given "another" partial map $h : A \rightharpoonup A$, equally satisfying these equations, namely:

$$\begin{aligned}&\mathrm{true}_A \mathrel{\widehat{=}} [\neg\, \chi \mathbin{\widehat{\circ}} a \implies h \mathbin{\widehat{\circ}} a \mathrel{\widehat{=}} a :] A \rightharpoonup \mathbb{2} \text{ (halt), and}\\ &\mathrm{true}_A \mathrel{\widehat{=}} [\,\chi \mathbin{\widehat{\circ}} a \implies h \mathbin{\widehat{\circ}} a \mathrel{\widehat{=}} h \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a\,] : A \rightharpoonup \mathbb{2}. \text{ (progress)}\end{aligned}$$

What we want to show is

$$\begin{aligned}&h \mathrel{\widehat{=}} \mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a \ =_{\text{by def}}\ f^{\S} \mathbin{\widehat{\circ}} (a, \mu \mathbin{\widehat{\circ}} a)\\ &\ =_{\text{by def}}\ f^{\S} \mathbin{\widehat{\circ}} (a, \mu\,[\,\neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S}\,] \mathbin{\widehat{\circ}} a) : A \rightharpoonup A.\end{aligned}$$

The **proof** of $h \mathrel{\widehat{=}} \mathrm{wh} : A \rightharpoonup A$ is by **dependent Peano Induction,** within (diagonal monoidal) p. r. theory $\widehat{\mathbf{S}}$ with its uniqueness $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$ of *initialised iterated*—inherited from basic, axiomatic version $(\mathrm{FR!})_{\mathbf{S}}$ for theory $\mathbf{S}$.

As—dependent—induction paramter we choose $m := \mu \mathbin{\widehat{\circ}} a : A \rightharpoonup \mathbb{N}$, abbreviation see above.

Dependent Peano **anchoring:**

$$m \doteq 0 \Rightarrow:\ \neg\, \chi \mathbin{\widehat{\circ}} a\ \wedge\ [\,h \mathbin{\widehat{\circ}} a\ \doteq_A\ a\,]\ \wedge\ [\,\mathrm{wh} \mathbin{\widehat{\circ}} a\ \doteq_A a\,] : A \times \mathbb{N} \rightharpoonup \mathbb{2}.$$

This **proves** the wanted *uniqueness* in the *anchor, i. e. halt* case $m \doteq 0$.

Dependent Peano **step** implication—to be shown now— is the following:

$$\begin{aligned} \mathrm{true}_{A^2 \times \mathbb{N}} \mathrel{\widehat{=}} [\, \mu \mathbin{\widehat{\circ}} a \ \doteq m \implies h \mathbin{\widehat{\circ}} a \ \doteq_A \ \mathrm{wh} \mathbin{\widehat{\circ}} a \,] \\ \Rightarrow [\, \mu \mathbin{\widehat{\circ}} a' \doteq s\, m \implies h \mathbin{\widehat{\circ}} a' \doteq_A \mathrm{wh} \mathbin{\widehat{\circ}} a' \,] : \\ A^2 \times \mathbb{N} \rightharpoonup 2. \end{aligned}$$

Conclusio of this asserted **step implication** follows from its **premise** via

$$\begin{aligned} & \mu \mathbin{\widehat{\circ}} a' \ \doteq \ s\, m \Rightarrow : \\ & \mu \ \widehat{\circ} f \mathbin{\widehat{\circ}} a' \ \doteq m \\ & \qquad \text{by bottom up characterisation of p. r. iteration} \\ & \qquad \text{and hence of the } \mu\text{-operator} \\ & \wedge \ h \mathbin{\widehat{\circ}} a' \ \doteq_A \ h \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a' \ \doteq_A \ \mathrm{wh} \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a' \doteq_A \mathrm{wh} \mathbin{\widehat{\circ}} a' : A \times \mathbb{N} \rightharpoonup 2, \end{aligned}$$

the latter line following from

$$\begin{aligned} & \mu \mathbin{\widehat{\circ}} a' \ \doteq \ s\, m \implies \chi \mathbin{\widehat{\circ}} a' \\ & \ \implies \mathrm{wh} \mathbin{\widehat{\circ}} a' \ \doteq_A \ \mathrm{wh} \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a' \\ & \qquad \wedge \ h \mathbin{\widehat{\circ}} a' \ \doteq_A \ h \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a' : A \times \mathbb{N} \rightharpoonup 2. \end{aligned}$$

A similar treatment **formalises** <u>until</u> loops: pseudocode for such

$$\mathrm{utl} = \underline{\mathrm{do}}\ f\ \underline{\mathrm{until}}\ \chi\ \underline{\mathrm{od}} \text{ is}$$

$$\begin{aligned}\mathrm{utl}(a) := [\, a' &:= f(a);\\ &\underline{\text{while}}\ \neg\,\chi(a')\ \underline{\text{do}}\ a' := f(a')\ \underline{\text{od}};\\ &\mathrm{utl}(a) := a'\,].\end{aligned}$$

Definition as a *partial* p. r. map:

$$\mathrm{u}tl\,[\,f : \chi\,](a) \ =_{\mathrm{def}}\ \mathrm{wh}\,[\,\neg\,\chi : f\,]\,\widehat{\circ}\,f : A \rightharpoonup A \rightharpoonup A.$$

This is already the general case: both f and χ possibly *partial.* Specialisation to **S**-maps f and χ looks similar: first factor $f : A \to A$ in **S**, second factor in general *partial,* in $\widehat{\mathbf{S}}$, despite of a—f and χ possibly assumed both to be in **S**.

Characterisation:

$$\begin{aligned}\chi\,\widehat{\circ}\,f\,\widehat{\circ}\,a &\implies \mathrm{utl}\,[\,f : \chi\,] \ \dot{=}_A\ f\,\widehat{\circ}\,a,\\ \neg\,\chi\,\widehat{\circ}\,f\,\widehat{\circ}\,a &\implies \mathrm{utl}\,[\,f : \chi\,]\,\widehat{\circ}\,a \ \dot{=}_A\ \mathrm{utl}\,[\,f : \chi\,]\,\widehat{\circ}\,f\,\widehat{\circ}\,a.\end{aligned}$$

Everything proven for wh above holds—mutatis mutandis—for until loops $\mathrm{utl}\,[\,f : \chi\,] : A \rightharpoonup A$.

With our "full" embedding of μ-recursive maps (over a theory **S**), into (categorical) theory $\widehat{\mathbf{S}}$ of partial **S**-maps, and the *converse result*—see Proof above, that each *partial* **S**-map—*morphism* in theory $\widehat{\mathbf{S}}$—has, within $\widehat{\mathbf{S}}$, a representation as a μ-recursive map "over" **S**, we arrive at

A Further Case of Church's Thesis:

- The *concept* of a partial p. r. map is equivalent to that of a μ-recursive (partial) map. It is another—free-variables, formally:

variable-free—notion of a *general recursive (partial) map,* all this in (and over) the categorical framework of an (arbitrary) *cartesian p. r. theory* **S** with (scheme of) abstraction of its predicates—as well as with equality *predicates* on those objects B which are common codomain of map pairs $f, g : A \to B$ taken into consideration, such that for these equality predicates $[\, b \doteq b' \,] : B^2 \to \mathbb{2}$ *Equality Definability* is guaranteed, main examples: all objects of $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$, and of Universe p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$.

This statement is slightly more general than the one(s) proven: Explicitely, we have considered just theories $\widehat{\mathbf{S}}$ extending specific theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ of (categorical) theory of primitive recursion with *predicate abstraction* and *their* extension by *partial maps.* But closer analysis of concepts and proofs shows that everything works for "basic" theory **S** taken a *cartesian* p. r. theory as just described.

- Same for <u>while</u> loops $\mathrm{wh} = \mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup \mathbb{2}$: They obviously *generate* all μ-recursive (partial) maps: For given (PR or partial PR) predicate $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$,

$$\mu\varphi \mathrel{\widehat{=}} r \mathbin{\widehat{\circ}} \mathrm{wh}\,[\,\neg\,\varphi : (A \times s)\,] : A \times \mathbb{N} \rightharpoonup A \times \mathbb{N} \to \mathbb{N}$$

  satisfies the characteristic implications for the μ-operator.

  Therefore the <u>while</u>-operator wh generates all *partial* maps, in $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$, even in just one step out of predicate/endo pairs $\chi : A \to \mathbb{2}$ and $f : A \to A$ in **S**.

- theory $\widehat{\mathbf{S}}$ is closed under the <u>while</u>-operator, as it is—and because it is—under the μ-operator.

- Formal Consequence of the last two assertions is in particular a fact known since long time to Computer Scientists: "one <u>while</u> loop is enough", starting from suitable <u>for</u> loop programs to define **S**-maps $\chi : A \to \mathbb{2}$ and $f : A \to A$, "data" for a <u>while</u> loop $\mathrm{wh}\,[\,\chi : f\,] : A \rightharpoonup A$.

  Since <u>for</u> loops—equivalent to p. r. maps—can in turn be written as (trivial) <u>while</u> loops,<u>while</u>-**Closure** of the fundamental maps: $0,\ s$, as well as substitutions—*logical functions* in the sense of Eilenberg & Elgot—reaches all of $\mu\mathbf{S}$, but presumably not in <u>while</u> nesting depth 1, as is the case when starting with all <u>for</u> loops, see above. I guess, for such a one-step closure by the <u>while</u>-operator, you need at least *case distinctions,* and these come in here—formally—as p. r. maps on their own right, namely as *induced* maps out of a *sum* $A \xrightarrow{i} A + B \xleftarrow{j} B$.

From a logical point of view, there are—at least—the following two open **Questions,** in

Arithmetics Complexity Problem:

- Does theory **PR** admit *strict, consistent* strengthenings, or is it a *simple theory,* will say that it admits its given notion of equality and the indiscrete (inconsistency) equality as only "congruences?", cf.a simple *group* which has as *normal subgroups* only itself and $\{1\}$. Because of reasons to be explained in later work, my guess here is: **PR** *admits* non-trivial strengthenings, in particular I suppose that the p. r. *trace* of **PA**, explained above, is a strict strengthening of **PR** resp. $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$.

We cannot exclude at present that all of these strengthening extensions of **PR** make up a whole lattice of (free-variables) arithmetical theories, each of them giving particular, "new" features to primitive recursive arithmetics.

- Already at start we possibly have such a strengthening: If free-variables ("free variables" in the classical sense) *primitive recursive arithmetic* **PRA** is defined to have as its terms all map terms obtainable by the (full) scheme of primitive recursion, and as formulae just the *defining equations* for the maps introduced by that scheme, then I see no way to prove all of the usual semiring equations for $\mathbb{N}$:

  We *need* Freyd's uniqueness (FR!) of the *initialised iterated:* From this HORN clause we can show (!) in particular GOODSTEIN's uniqueness rules U_1 to U_4 on which *his* Proof of the semiring properties of $\mathbb{N}$ is based. He takes these rules as *axioms.*

  My guess is here—if I have understood right the definition of **PRA**—that $\mathbf{PR} = \mathbf{PR}+(\mathrm{FR!})$ is a strict strengthening of **PRA**, at least if there is no "underground" connection to the set theoretic view of maps as (possibly infinite) *argument-value tables.*

- Finally, *descent* theory $\pi\mathbf{R}$ in chapter 3 below—defined by axiom of *non-infinite iterative descent*—presumably is a strict strengthening of theory **PRa**. It is not excluded that theory $\pi\mathbf{R}$ is *simple* in the sense above, or can at least be strengthened into a simple theory by a stronger descent axiom as for example iterative non-infinite descent in ordinal $\mathbb{U}$ of *nested strings.*

Note

The concept of a monoidal category with (symmetry- and in addition) diagonal and terminal substitution transformations goes back to Budach & Hoehnke 1975. They observe that the category of sets and partial maps is a diagonal symmetric half-terminal category. *Half*-terminal, since the "terminal" maps $\Pi_A : A \to \mathbb{1}$ no longer constitute a *natural* transformation.

Chapter 4

Universal Sets and Universe Theories

Within theory **PRa** of primitive recursion with predicate abstraction we construct (in a categorical way) a *universal object* X of all *nested pairs of natural numbers* in which all objects of **PRa**—subobjects of finite powers of NNO $\mathbb{N}$—are embedded. This gives rise to the theories $\mathbf{PR}\mathbb{X} \sqsubset \mathbf{PRa}$ of **PR** augmented by universal object $\mathbb{X}$, and $\mathbf{PR}\mathbb{X}\mathbf{a}$, namely $\mathbf{PR}\mathbb{X}$ with predicate abstraction which has all properties wanted, see **universal embedding theorem.** These two theories will be basic an for the logical chapters to come, on evaluation, soundness, decision, and consistency.

4.1 Strings as polynomials

Strings $a_0\, a_1\, \ldots\, a_n$ of natural numbers (in set $\mathbb{N}^+ = \mathbb{N}^* \smallsetminus \{\Box\}$ of non-empty strings) are coded as *prime power products*

$$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_{>0} \subset \mathbb{N},\ p_j \text{ the } j\,\text{th prime number.}$$

Formally: euclidean prime power factorisation gives rise to a p. r. *projection* family

$$\pi = \pi_j(a) : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N},\ a = p_0^{\pi_0(a)} \cdot p_1^{\pi_1(a)} \cdot \ldots \cdot p_a^{\pi_a(a)},$$

unique $\pi_j(a)$, $\pi_j(a) = 0$ for $j > n,\ n = n(a) : \mathbb{N}_> \to \mathbb{N}$ suitable p. r.

Strings $a_0\, a_1\, \ldots\, a_n \equiv p_0^{a_0} \cdot \ldots \cdot p_n^{a_n}$ are identified with (the coefficient lists of) "their" *polynomials*

$$p(X) = a_0 + a_1 X^1 + \ldots + a_n X^n \text{ as well as}$$
$$p(\omega) = a_0 + a_1 \omega^1 + \ldots + a_n \omega^n,$$

in *indeterminate* X resp. ω.

Componentwise addition (and truncated subtraction), as well as

$$p(\omega) \cdot \omega = \sum_{j=0}^{n} a_j \omega^{j+1} \equiv \prod_{j=0}^{n} p_{j+1}^{a_j},$$

special case of Cauchy product of polynomials.

Reverse-lexicographical Order of NNO strings and polynomials: order priority of (coefficient-)strings from right to left, has—intuitively, and formally within **sets**—only *finite descending chains.*

This applies in particular to descending complexities of CCI's: *Complexity Controlled Iterations* below, with complexity values in $\mathbb{N}[\omega]$; p. r. map code *evaluation* will be resolved into such a CCI.

4.2 Universal object $\mathbb{X}$ of numerals and nested pairs

We begin the construction of Universal object by internal *numeralisation* of all objective natural numbers, of objective numerals

$$\begin{aligned}
&\mathrm{num}(0) \equiv 0 : \mathbb{1} \to \mathbb{N}, \\
&\mathrm{num}(1) \equiv 1 \ =_{\mathrm{def}} \ (s(0)) : \mathbb{1} \to \mathbb{N} \to \mathbb{N}, \\
&\mathrm{num}(2) \equiv 2 \ =_{\mathrm{def}} \ (s(s(0)) : \mathbb{1} \to \mathbb{N} \\
&\mathrm{num}(\underline{n}+1) \equiv \underline{n}+1 \ =_{\mathrm{def}} \ (s(\underline{n})) : \mathbb{1} \to \mathbb{N}, \\
&\underline{n} \in \underline{\mathbb{N}} \text{ meta-variable.}
\end{aligned}$$

Internal numerals, *numeralisation*

$$\nu = \nu(n) : \mathbb{N} \to \mathbb{N}^+ \equiv \mathbb{N}^* \setminus \{0\} \equiv \mathbb{N}_> \subset \mathbb{N} :$$

$$\begin{aligned}
&\nu(0) \ =_{\mathrm{def}} \ \ulcorner 0 \urcorner : \mathbb{1} \to \mathbb{N} \text{ code (}\textit{goedel number}\text{) of } 0, \\
&\nu(1) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(0) \rangle \ =_{\mathrm{by\,def}} \ \langle \ulcorner s \urcorner \ulcorner \circ \urcorner \ulcorner 0 \urcorner \rangle : \mathbb{1} \to \mathbb{N},
\end{aligned}$$

abbreviation for (string) goedelisation, here in particular for **LaTeX** source code

$$\begin{aligned}
&\ulcorner (\urcorner \ulcorner s \urcorner \ulcorner \circ \urcorner \nu(0) \ulcorner) \urcorner = \ulcorner (\urcorner \ulcorner s \urcorner \ulcorner \circ \urcorner \ulcorner 0 \urcorner \ulcorner) \urcorner \\
&\equiv p_0^{\mathbf{ASCII}[(]} \ p_1^{\mathbf{ASCII}[s]} \ p_2^{\mathbf{ASCII}[\backslash\mathrm{circ}]} \ p_3^{\mathbf{ASCII}[0]} \ p_4^{\mathbf{ASCII}[)]} \\
&\equiv 2^{40} \ 3^{115} \ 5^{\mathbf{ASCII}[\backslash\mathrm{circ}]} \ 7^{48} \ 11^{41} : \mathbb{1} \to \mathbb{N}, \\
&\text{an element of } \underline{\mathbb{N}}, \text{ a } \textit{constant} \text{ of } \mathbb{N},
\end{aligned}$$

$\nu(2) \;=_{\text{def}}\; \langle \ulcorner s \urcorner \odot \nu(1) \rangle = \langle \ulcorner s \urcorner \odot \langle \ulcorner s \urcorner \odot \nu(0) \rangle \rangle$ etc. PR:
$\nu(n+1) \;=_{\text{def}}\; \langle \ulcorner s \urcorner \odot \nu(n) \rangle \in \mathbb{N}.$
$\nu(n)$ has n closing brackets (at end).

This internal numeralisation distributes the "elements", numbers of the NNO $\mathbb{N}$, with suitable gaps over $\mathbb{N}$: the gaps then will receive in particular codes of any other symbols of object Languages **PR** and **PRa** as well as of Universe Languages **PR**$\mathbb{X}$ and **PR**$\mathbb{X}$**a** to come.

ν-**Predicate lemma:** Enumeration $\nu : \mathbb{N} \to \mathbb{N}$ defines a characteristic predicate $\mathrm{im}[\nu] = \chi_\nu : \mathbb{N} \to \mathbb{2}$, and by this object

$$\nu\mathbb{N} = \{\mathbb{N} : \chi_\nu\} \subset \mathbb{N}^+$$

of internal numerals $\nu\mathbb{N} \cong \mathbb{N}$.

Proof: Use finite $\exists$—iterative '$\vee$'—for definition of $\mathrm{im}[\nu]$, as follows:

$$\chi_\nu(c) \;=_{\text{def}}\; \vee_{n \le c}[c \doteq \nu(n)]$$
$$= [c \doteq \nu(0) \,\vee\, c \doteq \nu(1) \,\vee\, \ldots \,\vee\, c \doteq \nu(c)] : \mathbb{N} \to \mathbb{2} \quad \textbf{q. e. d.}$$

$\nu : \mathbb{N} \to \mathbb{N}^+ \subset \mathbb{N}$ has codomain restriction

$\nu : \mathbb{N} \to \nu\mathbb{N} \;=_{\text{def}}\; \{\mathbb{N} : \chi_\nu\}$
and is then an iso with p. r. inverse
$\nu^{-1} = \nu^{-1}(c) \;=_{\text{def}}\; \min\limits_{n \le c}[\nu(n) \doteq c] : \nu\mathbb{N} \xrightarrow{\cong} \mathbb{N}.$

For a **PR**-map $f : \mathbb{N} \to \mathbb{N}$ **define** its *numeral twin*

$$\dot{f} \;=_{\text{def}}\; \nu \circ f \circ \nu^{-1} : \nu\mathbb{N} \xrightarrow{\nu^{-1}} \mathbb{N} \xrightarrow{f} \mathbb{N} \xrightarrow{\nu} \nu\mathbb{N},$$

giving trivially (local) *naturality*

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[d, "\cong"', "\nu"] \arrow[dr, phantom, "="] & \mathbb{N} \arrow[d, "\cong"', "\nu"] \\ \nu\mathbb{N} \arrow[r, "\dot{f}"] \arrow[u, bend left, "\nu^{-1}"] & \nu\mathbb{N} \end{tikzcd}$$

Extension of numeral sets and numeralisation to all objects of **PR** (and of **PRa** :)

- $\nu\mathbb{1} = \{\nu 0\} = \{\ulcorner 0 \urcorner\} \subset \nu\mathbb{N} \subset \mathbb{N}$,
  $\nu_{\mathbb{1}}(0) = \nu(0) : \mathbb{1} \xrightarrow{\cong} \nu\mathbb{1} \xrightarrow{\subset} \nu\mathbb{N}$.
- recursive extension to products:

$$\frac{A, B \text{ in } \mathbf{PR}}{\begin{array}{l} \nu(A \times B) = \langle \nu A \dot{\times} \nu B \rangle \\ =_{\text{def}} \ \{\langle \nu A(a); \nu B(b) \rangle : a \in A, b \in B\} \\ \text{predicatively} \\ = \{\langle c; d \rangle \in \mathbb{N} : \chi_{\nu A}(c) \wedge \chi_{\nu B}(d)\}. \end{array}}$$

- Extension to (predicative) subsets:

$$\frac{\chi = \chi(a) : A \to \mathbb{N} \text{ predicate}}{\nu\{A : \chi\} \ =_{\text{def}} \ \{\nu(a) : a \in \{A : \chi\}\} \subseteq \nu A}$$

- **remark:** $\mathbb{X}$, $\nu\mathbb{X} \subset \mathbb{N}$, $\nu\mathbb{X} \cong \mathbb{X}$, but $\nu X \subsetneq \mathbb{X}$, parallel to $\nu\mathbb{N} \subsetneq \mathbb{N}$.

- ν isomorphy (and *naturality*) extend to A, B in **PR** and in **PRa**.

Universal objects $\mathbb{X}$, $\mathbb{X}_{\underline{\perp}}$ of numerals and (nested) pairs of numerals:

As code for *waste symbol* we take
$\underline{\perp} \ =_{\mathrm{def}} \ulcorner\perp\urcorner \equiv \ulcorner\backslash\mathrm{bot}\urcorner : \mathbb{1} \to \mathbb{N}.$

Define sets

$$\mathbb{X}, \mathbb{X}_{\underline{\perp}} = \{\mathbb{N} : \mathbb{X}, \mathbb{X}_{\underline{\perp}} : \mathbb{N} \to \mathbb{2}\} \subset \mathbb{N}$$

of all (codes of)

- *undefined value* $\underline{\perp}$,

- *numerals* $\nu(n) \in \nu\mathbb{N}$, and

- (possibly nested) *pairs*
  $\langle x; y\rangle \ =_{\text{by def}} \ \ulcorner(\urcorner x\ulcorner, \urcorner y\ulcorner)\urcorner$ of numerals

as follows:

- $\nu\mathbb{N} \subset \mathbb{X} \subset \mathbb{N}$, *numerals proper;* further recursively enumerated:

- $\langle \mathbb{X} \dot{\times} \mathbb{X}\rangle \ =_{\mathrm{def}} \ \{\langle x; y\rangle : x, y \in \mathbb{X}\} \subset \mathbb{X}$,

  set of *(nested) pairs of numerals, general numerals,* in particular

$$\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle = \{\langle x; \nu n\rangle : x \in \mathbb{X}, n \in \mathbb{N}\} \subset \mathbb{X};$$

- $\mathbb{X}_{\perp} =_{\text{def}} \mathbb{X} \cup \{\perp\} \subset \mathbb{N}^+$.

$\mathbb{X}$-**Predicative Lemma:** $\mathbb{X}$ has predicative form

$$\mathbb{X} = \{\mathbb{N} : \chi_{\mathbb{X}}\}, \text{ and } \mathbb{X}_{\perp} = \{\mathbb{N} : \chi_{\mathbb{X}} \vee \{\ulcorner \perp \urcorner\}\}.$$

Proof as (technically advanced) **Exercise.**

This terminates recursive **definition** of ("minimal") predicative *Universal objects* $\mathbb{X}$ and $\mathbb{X}_{\perp}$, of *nested pairs of numerals,* both

$$\mathbb{X},\ \mathbb{X}_{\perp} \subset \mathbb{N}^+ \equiv \mathbb{N}_> =_{\text{by def}} \mathbb{N}_{>0} \subset \mathbb{N} \equiv \mathbb{N}^*.$$

Remark: A *superUniversal object* $\mathbb{U} \supset \mathbb{X}$, $\mathbb{U} \subset \mathbb{N}$ of *lists* (bracketed strings) of numerals can be defined p. r. by

- $\nu\mathbb{N} \subseteq \mathbb{U}$,
- $x \in \mathbb{U}, y \in \mathbb{U} \implies x; y \in \mathbb{U}$,
- $x \in \mathbb{U} \implies \langle x \rangle \in \mathbb{U}$.

(Predicative) set $\mathbb{U} \subset \mathbb{N}$ can be interpreted as set of (numeralised) coefficient lists $\mathbb{N}[X_1, X_2, \ldots, X_m, \ldots]$ of polynomials in *several indeterminates* $X_1, X_2, \ldots$ with (numeralised) coefficients out of $\nu\mathbb{N}$, written in form $\cup_m \mathbb{N}[X_1][X_2] \ldots [X_m]$.

4.3 Universe monoid PR$\mathbb{X}$

The endomorphism set $\mathbf{PR}(\mathbb{N}, \mathbb{N}) \subset \mathbf{PR}$ is itself a monoid, a categorical theory with just one object.

Embedded "cartesian p. r. Monoid" **PR**$\mathbb{X}$:

- the basic, "super" object of $\mathbf{PR}\mathbb{X}$ is
  $\mathbb{X}_{\underline{\perp}} = \mathbb{X} \,\dot{\cup}\, \{\underline{\perp}\} = \mathbb{X} \,\dot{\cup}\, \{\ulcorner\perp\urcorner\} \subset \mathbb{N}$,
  $\mathbb{X} : \mathbb{N} \to \mathbb{N}$ in $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ predicate/set of (internal) numerals and nested pairs of numerals.

- the rôle of the NNO will be taken by the above predicative subset
  $$\nu\mathbb{N} = \{c \in \mathbb{N} : \chi_\nu(c)\} \subset \mathbb{X} \subset \mathbb{X}_{\underline{\perp}} \subset \mathbb{N}$$
  of the internal *numerals.*

- the basic "universe" map constants of $\mathbf{PR}\mathbb{X}$,
  ba $\in$ bas set of those maps, are
  - "identity" $\mathring{\mathrm{id}} = \mathrm{id}_{\mathbb{X}} : \mathbb{N} \supset \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \to \mathbb{X} \subset \mathbb{X}_{\underline{\perp}}$,
    $\mathbb{X} \ni x \mapsto x \in \mathbb{X}$,
    $\mathbb{N} \setminus \mathbb{X} \ni z \mapsto \underline{\perp}$ (***trash***),
    PR map code set "from" $\mathbb{N}$ "to" $\mathbb{N}$, same for all codes below.
  - "zero" (redefined for $\mathbf{PR}\mathbb{X}$) $\mathring{0} : \mathbb{X} \to \mathbb{X}_{\underline{\perp}}$,
    $\mathbb{X} \ni \nu 0 \mapsto \nu 0 \in \nu\mathbb{N} \subset \mathbb{X}$,
    $\mathbb{N} \setminus \{\nu 0\} \ni z \mapsto \underline{\perp}$,
  - "successor" $\mathring{s} : \mathbb{X}_{\underline{\perp}} \to \mathbb{X}_{\underline{\perp}}$:
    $\nu n \mapsto \nu(s\,n) \; =_{\text{by def}} \; \langle \ulcorner s \urcorner \odot \nu(n) \rangle$,
    $\mathbb{N} \setminus \nu\mathbb{N} \ni z \mapsto \underline{\perp}$.
  - "terminal map": $\mathring{\Pi} : \mathbb{X} \to \nu\mathbb{1} \subset \mathbb{X}$,
    $\mathbb{X} \ni x \mapsto \nu 0 \in \nu\mathbb{1} = \{\nu 0\} \subset \mathbb{X}$,
    $\mathbb{N} \setminus \mathbb{X} \ni z \mapsto \underline{\perp}$.

- "left projection":
  $\mathring{\ell} : \mathbb{N} \supset \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \mathbb{X}_{\perp}$,
  $\langle x; y \rangle \mapsto x \in \mathbb{X}$, $\nu\mathbb{N} \ni \nu n \mapsto \underline{\perp}$, $\underline{\perp} \mapsto \underline{\perp}$.
- "right projection" $r \in$ bas analogous.

- close Monoid **PR**$\mathbb{X}$ under composition of theory **PR** :

$$(\circ) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})}{(g \circ f) \text{ in } \mathbf{PR}\mathbb{X},}$$

trash propagation clear.

- "induced map":

$$(\text{ind}) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X}}{\langle f \,.\, g \rangle \text{ in } \mathbf{PR}\mathbb{X}, \text{ defined by}}$$

$\mathbb{X} \ni x \mapsto \langle f\,x; g\,x \rangle \in \mathbb{X}$.

- "product map":

$$(\dot{\times}) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X}}{\langle f \dot{\times} g \rangle \text{ in } \mathbf{PR}\mathbb{X}, \text{ defined by}}$$

$\mathbb{X} \ni \langle x; y \rangle \mapsto \langle f\,x; g\,y \rangle \in \mathbb{X}$,

$\mathbb{N} \setminus \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \ni z \mapsto \underline{\perp}$.

- "iterated" (formally interesting, see last lines):

$$(\mathrm{it})\quad \frac{f : \mathbb{X} \to \mathbb{X}\ \mathbf{PR}\mathbb{X}\ \text{map, in particular}\ \underline{\perp} \mapsto \underline{\perp}}{\begin{array}{l} f^{\dot{\S}} : \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle \to \mathbb{X}\ \text{in}\ \mathbf{PR}\mathbb{X}, \\ \langle x; \dot{n} \rangle \mapsto f^n(x) \in \mathbb{X}, \\ n = \nu^{-1}(\dot{n}),\ \dot{n} \in \dot{\mathbb{N}} = \nu\mathbb{N} \ =_{\text{by def}}\ \{\mathbb{N} : \chi_\nu\}\ \text{free}, \\ \mathbb{N} \ni z \mapsto \underline{\perp}\ \text{for}\ z\ \text{not of form}\ \langle x; \dot{n} \rangle. \end{array}}$$

[Predicates $\nu\mathbb{N}$ and $\langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle : \mathbb{N} \to \mathbb{N}$ work as auxiliary objects, subobjects of $\mathbb{X} : \mathbb{N} \to \mathbb{N}$.]

- Notion of map equality for theory $\mathbf{PR}\mathbb{X}$ is inherited(!) from $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ i. e. from theory $\mathbf{PR}$.

$\mathbf{PR}\mathbb{X}$ Structure theorem: With emerging (predicative) objects $\mathbb{X}, \nu\mathbb{1}, \nu\mathbb{N}$,

$$\frac{A, B\ \text{objects}}{\langle A \dot{\times} B \rangle\ \text{object},}$$

constants, maps, composition above,

- $\nu\mathbb{1} = \{\nu 0\}$ taken as "terminal object",
- $\mathring{\Pi} : \mathbb{X} \to \nu\mathbb{1}$ taken as "terminal map,"

- "Product" taken

$$\langle \mathring{\ell} : \langle A \dot{\times} B \rangle \to A : \langle x; y \rangle \to x,$$
$$\mathring{r} : \langle A \dot{\times} B \rangle \to B, \langle x; y \rangle \to y \rangle,$$

- $\langle f \,.\, g \rangle : C \to \langle A \dot{\times} B \rangle$, $x \mapsto \langle f\, x; g\, x \rangle$,
  taken as "induced map,"

- $\langle f \dot{\times} g \rangle : \langle A \dot{\times} B \rangle \to \langle A' \dot{\times} B' \rangle$, $\langle x; y \rangle \mapsto \langle f\, x; g\, y \rangle$,
  taken as "map product,"

- $\langle \nu \mathbb{1} \xrightarrow{\mathring{0}} \nu \mathrm{N} \xrightarrow{\mathring{s}} \nu \mathrm{N} \rangle$ taken as NNO,

- and $f^{\dot{\S}} : \langle \mathbb{X} \dot{\times} \nu \mathrm{N} \rangle \to \mathbb{X}$ as iterated of
  PR$\mathbb{X}$ endomap $f : \mathbb{X} \to \mathbb{X}$, $\langle x; \nu n \rangle \mapsto f^n(x) = f^{\S}(x, n)$,

PR$\mathbb{X}$ becomes a cartesian p. r. category with universal object.

• Fundamental theory **PR** is naturally embedded into theory **PR**$\mathbb{X}$, by faithful functor **I** say.

4.4 Typed universe theory PR$\mathbb{X}$a

Let emerge within universe monoid/universe cartesian p. r. theory all **PRa** objects $\{A : \chi\}$ as additional objects $\nu\{A : \chi\}$ and get this way a p. r. cartesian theory **PR$\mathbb{X}$a** with extensions of predicates, finite limits, finite sums, coequalisers of equivalence predicates, as well as with (formal, "including") universal object $\mathbb{X}$, of numerals and (nested) pairs of numerals.

Universal embedding theorem:

(i) $\mathbf{I}: \mathbf{PR} \longrightarrow \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$ above is a faithful functor .

(ii) theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ "inherits" from category $\mathbf{PRa}$ all of its (categorically described) structure: cartesian p. r. category structure, equality predicates on all objects, scheme of predicate abstraction, equalisers, and—trivially—the whole algebraic, logic and order structure on NNO $\nu\mathbb{N}$ and truth object $\nu 2$.

(iii) $\mathbf{PR}$ map embedding $\mathbf{I}$ "canonically" extends into a cartesian p. r. functorial embedding (!)

$$\mathbf{I}: \mathbf{PRa} \longrightarrow \mathbf{PR}\mathbb{X}\mathbf{a} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$$

of theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ into emerging *universe theory* $\mathbf{PR}\mathbb{X}\mathbf{a}$ *with predicate abstraction.*

(iv) Embedding $\mathbf{I}$ defines a <u>p. r.</u> isomorphism of categories

$$\mathbf{I}: \mathbf{PRa} \xrightarrow{\cong} \mathbf{I}[\mathbf{PRa}] \sqsubset \mathbf{PR}\mathbb{X}\mathbf{a}.$$

(v) (internal) code set is

$\lceil \mathbb{X}, \mathbb{X} \rceil =_{\text{by def}} \lceil \mathbb{X}, \mathbb{X} \rceil_{\mathbf{PR}\mathbb{X}\mathbf{a}} = \lceil \mathbb{X}, \mathbb{X} \rceil_{\mathbf{PR}\mathbb{X}} = \mathrm{PR}\mathbb{X}.$

Internal notion $\check{=}$ of equality is in both cases inherited from internal notion of equality of theories $\mathbf{PR}$, $\mathbf{PR}(\mathbb{N}, \mathbb{N})$, given as enumeration of internally equal pairs

$\check{=} \; = \; \check{=}_k : \mathbb{N} \to \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X} \subset \mathbb{N} \times \mathbb{N},$

as well as predicatively as

$\check{=} \; = \; u \check{=}_k v : \mathbb{N} \times (\mathrm{PR} \times \mathrm{PR}) \to 2 :$

kth *internal equality instance equals pair* (u, v) *of internal maps.*

(vi) put things together into the following diagram:

$$\begin{tikzcd}
 & \{A:\chi\} \arrow[r, "f"] \arrow[d, "\nu\{A:\chi\}"', "\cong"] \arrow[dr, phantom, "="] & \{B:\varphi\} \arrow[d, "\cong"', "\nu\{B:\varphi\}"] & \\
\nu\{A:\chi\} \arrow[r, equal] & \mathbf{I}\,\{A:\chi\} \arrow[r, "\mathbf{I}\,f"] \arrow[d, "\subset"] & \mathbf{I}\,\{B:\varphi\} \arrow[r, "\subset"] & \mathbf{I}\,\{B:\varphi\}\,\dot{\cup}\,\{\underline{\perp}\} \arrow[d, "\subset"] \\
 & \mathbb{X}_{\underline{\perp}} \arrow[rr, "\dot{f}\ =_{\text{by def}}\ \mathbf{I}_{\mathbf{PR}}\,f"] \arrow[d, "\subset"] \arrow[drr, phantom, "="] & & \mathbb{X}_{\underline{\perp}} \arrow[d, "\subset"] \\
 & \mathbb{N} \arrow[rr, "\dot{f}"] & & \mathbb{N}
\end{tikzcd}$$

PRa embedding DIAGRAM for $\mathbf{I}\,f$ **q. e. d.**

Chapter 5

Evaluation of p. r. map codes

Double recursive evaluation of p. r. map codes, and arguments in universal set $\mathbb{X}$ of nested pairs of natural numbers, can be resolved into an iteration of elementary evaluation steps on code/argument pairs. Such a step evaluates the actual basic code particle on the actual argument and simplifies the code accordingly. Each step diminishes a suitably defined complexity out of the hierarchically ordered semiring $\mathbb{N}[\omega]$ of polynomials until complexity 0 and the code/argument pair $(\ulcorner \mathrm{id} \urcorner, \text{evaluation result})$ is reached.

Evaluation, defined as this *complexity controlled iteration,* satisfies in fact the characteristic double recursive equations, and evaluates concrete map codes $\ulcorner f \urcorner$ into $ev(\ulcorner f \urcorner, a) = f(a)$ (*objectivity*).

Descending complexity of codes is introduced for to make sure termination—intuitively in $\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}$ and formally in **set** theory $\mathbf{T}$. A theory-strengthening $\pi\mathbf{R}$ of $\mathbf{PR}\mathbb{X}\mathbf{a}$ is introduced by an additional axiom (π) stating the impossibility of infinitely descending complexity controlled iterations, the way we arrive to circumscribe termination within our constructive context. On this theory $\pi\mathbf{R}$ will bear our positive assertions about *evaluation soundness* and consistency.

5.1 Complexity controlled iteration

The data of such a **CCI** are an endomap $p = p(a) : A \to A$ (*predecessor*), and a *complexity* map $c = c(a) : A \to \mathbb{N}[\omega]$ on p's domain. Complexity *values* are taken in reverse-lexicographically ordered polynomial object $\mathbb{N}[\omega] \equiv \mathbb{N}^+ \equiv \mathbb{N}^* \smallsetminus \{\Box\} \equiv \mathbb{N}_>$.

Definition: $[c : A \to \mathbb{N}[\omega], p : A \to A]$ constitute the data of a *Complexity Controlled Iteration* $\mathrm{CCI} = \mathrm{CCI}[c, p]$, if

- $(a \in A)[c(a) > 0 \implies c\,p(a) < c(a)]$ *(descent)*

  as well as, for commodity,

- $(a \in A)[c(a) \doteq 0 \implies p(a) \doteq a]$ *(stationarity).*

Such data define a <u>while</u> loop

$\mathrm{wh}[c > 0, p] : A \rightharpoonup A,$ more explicetly written
<u>while</u> $c(a) > 0$ <u>do</u> $a := p(a)$ <u>od</u>.

We rely on the following **axiom scheme** of *non-infinite iterative* *descent:*

$$
\begin{array}{ll}
& \mathrm{CCI}[c = c(a) : A \to \mathbb{N}[\omega],\ p = p(a) : A \to A] : \\
& c, p \text{ make up a } \textit{complexity controlled iteration:} \\
& c = c\,(a) : A \to \mathbb{N}[\omega] \text{ p.r. } (\textit{complexity}), \\
& p = p(a) : A \to A \text{ p.r. } (\textit{predecessor endo}), \\
& c\,(a) > 0 \implies c\,p(a) < c\,(a)\ (\textit{descent}), \\
& c\,(a) \doteq 0 \implies p(a) \doteq a\ (\textit{stationarity}) \\
& \psi = \psi(a) : A \to \mathbb{2}\ \textit{“negative” test predicate:} \\
& [\psi(a) \implies c\,p^n(a) > 0],\ a \in A, n \in \mathbb{N} \text{ both free,} \\
& \quad (\text{“all } n\text{”, } \textit{to be excluded}) \\
(\pi) & \overline{} \\
& \psi(a) = \mathrm{false}_A(a) : A \to \mathbb{2}.
\end{array}
$$

The first six lines of the *antecedent* constitute (p, c) as a CCI : a *Complexity Controlled Iteration,* with (stepwise) descending order values in—polynomial—*ordinal* $\mathbb{N}[\omega] \subset \mathbb{N}^*$ ordered reverse-lexicographically.

The scheme says: *A predicate ψ which implies a CCI to (overall)* infinitely *descend must be (overall) false.*

By contraposition this can be turned into

$$
(\pi^+)\quad \frac{\begin{array}{l} c, p \text{ define a CCI,} \\ \varphi = \varphi(a) : A \to \mathbb{2} \text{ \textit{“positive” test predicate:}} \\ [c\, p^n(a) \doteq 0 \implies \varphi(a)] : A \times \mathbb{N} \to \mathbb{2} \\ \quad (\text{“exists } n\text{”, \textit{to be asserted}}) \end{array}}{\varphi(a) = \mathrm{true}_A(a) : A \to \mathbb{2}.}
$$

A predicate which holds under the premise of termination of a CCI must be true by itself. This is to express that a CCI must terminate anyway. It says that the *defined arguments enumeration* of a CCI considered as a while loop is a p. r. epimorphism (not a retraction in general.) Technically, we will rely on the (negative) form (π) of the axiom.

- central **example:** *general recursive,* Ackermann type *PR-code evaluation ev* to be *resolved* into such a CCI.

- scheme (π) is a theorem for set theory $\mathbf{T}$ with its quantifiers $\exists$ and $\forall$, and with its having $\mathbb{N}[\omega] \equiv \omega^\omega$ as a (countable) *ordinal:* existential guarantee of finiteness of descending chains within ω^ω.

- without quantification, namely for theories like $\mathbf{PRa}, \mathbf{PR}\mathbb{X}\mathbf{a}$, we are lead to this inference-of-equations scheme guaranteeing (intuitively) termination of CCIs, in particular termination of iterative p. r. code evaluation.

Comment: The point is that (π) expresses an axiom which "we all" believe in (and which is a theorem in **set** theory): Nobody has pointed to—will be able (?) to point to—any *infinitly descending chain* in $\mathbb{N}[\omega] \ =_{\text{by def}} \ \mathbb{N}^+ \subset \mathbb{N}^*$ (provided with its reverse-lexicographical order)—a fortiori *not* to any *iterative* such—to any infinitly descending CCI. For **set** theory (if consistent,) this is in fact impossible, by

set-(π) **Lemma: set** theory satisfies descent scheme (π).

Proof (asked for by J. Wleczyk): For $c = c_n : \mathbb{N} \to \mathbb{N}[\omega]$ a chain:

$$\begin{aligned} &\forall n[[c_n > 0 \Rightarrow c_{n+1} < c_n] \wedge [c_n = 0 \Rightarrow c_{n+1} = 0]] \text{ (descent)} \\ &\implies \exists n_0 \, (\forall n > n_0)[c_n = 0\,\omega^0] \text{ (termination).} \end{aligned} \tag{$\bullet$}$$

Proof of $(\bullet)$ by **nested induction** on degree $m = \deg(c_0)$ of first member $c_0 = \sum_{i=0}^m c_{0i}\omega^i$ of chain $c_0 \in \mathbb{N}[\omega]$:

$m = 0 : n_0 = c_{00}$;

assume given n_m such that $c_{m_n} = 0\,\omega^0,$ then choose

$n_{m+1} = n_m + c_{m0}$ to do the job for c_{m+1} :

induction on $n_m \in \mathbb{N}$:

$n_m = 0, \ c_{n0} = 0 : \ n_{m+1} = n_m + 1$ does its job, since here

$$c_{m0} = 0 \text{ or } \deg c_1 = \deg \sum_{i=0}^m c_{1i} \leq m, \text{whence } c_{n_{m+1}} = 0\,\omega;$$

induction step $n_m \mapsto n_{m+1}$: choose $n_{m+1} = n_m + 1 + c_{m+1\,0}$,

then in fact $c_{n_{m+1}} = c_{n_m + c_{0\,m+1}} = 0\,\omega$ **q. e. d.**

Definition: Call *PR descent theory* universe theory $\pi\mathbf{R} \ =_{\text{def}}$ $\mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$ strengthened by axiom scheme (π) above of non-infinite descent.

5.2 PR code set

The *map code set*—set of gödel numbers—we want to *evaluate* is $\mathrm{PR}\mathbb{X} = \lceil\mathbb{X}, \mathbb{X}\rceil \subset \mathbb{N}$. It is p. r. **defined** as follows:

- $\ulcorner \mathrm{ba} \urcorner \in \mathrm{PR}\mathbb{X}$—formal categorically:

  $\mathrm{PR}\mathbb{X} \circ \ulcorner \mathrm{ba} \urcorner = \mathrm{true}$—this for basic map constant

  $\mathrm{ba} \in \mathrm{bas} = \{\mathring{0}, \mathring{s}, \mathring{\mathrm{id}}, \mathring{\Pi}, \mathring{\Delta}, \mathring{\ell}, \mathring{r}\}$: *zero, successor, identity, terminal map, diagonal, left and right projection.* All of these interpreted into endo map Monoid $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$ of fundamental cartesian p. r. theory **PR**.

- for u, v in $\mathrm{PR}\mathbb{X}$ in general add

  - internally *composed:* $\langle v \odot u \rangle = \ulcorner (\urcorner v \ulcorner \circ \urcorner u \ulcorner) \urcorner :$

    $\mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X} \to \mathrm{PR}\mathbb{X}$, $u, v \in \mathrm{PR}\mathbb{X}$ both free,

    in particular $\ulcorner (g \circ f) \urcorner = \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \in \mathrm{PR}\mathbb{X}$

    for $f, g : \mathbb{X} \to \mathbb{X}$ in $\mathbf{PR}\mathbb{X}$;

  - internally *induced:* $\langle u; v \rangle = \ulcorner (\urcorner u \ulcorner, \urcorner v \ulcorner) \urcorner \in \mathrm{PR}\mathbb{X}$,

    in particular $\ulcorner (f, g) \urcorner = \langle \ulcorner f \urcorner . \ulcorner g \urcorner \rangle \in \mathrm{PR}\mathbb{X}$;

  - internal *cartesian product:* $\langle u \# v \rangle \in \mathrm{PR}\mathbb{X}$,

    $u, v \in \mathrm{PR}\mathbb{X}$ free, in particular

    $\ulcorner (f \dot{\times} g) \urcorner = \langle \ulcorner f \urcorner \# \ulcorner g \urcorner \rangle \in \mathrm{PR}\mathbb{X}$;

  - internally *iterated:* $u^{\$} = u^{\ulcorner \dot{\S} \urcorner} \in \mathrm{PR}\mathbb{X}$, $u \in \mathrm{PR}\mathbb{X}$,

    in particular $\ulcorner f^{\dot{\S}} \urcorner = \ulcorner f \urcorner^{\$} \in \mathrm{PR}\mathbb{X}$.

5.3 Iterative evaluation

For **Definition** of *evaluation ev* we first introduce *evaluation step* of form

$$e\,(u,x) = (e_{\mathrm{map}}(u,x), e_{\mathrm{arg}}(u,x)) : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}\,,$$

by primitive recursion. This within "outer" theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ which already has **PR** predicates $\mathbb{X}, \mathbb{X}_{\underline{\perp}} \ =_{\mathrm{by\,def}}\ \mathbb{X} \cup \{\underline{\perp}\} = \mathbb{X} \cup \{\ulcorner\perp\urcorner\}$, and $\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle : \mathbb{N} \to \mathbb{N}$ as objects.

Comment: $e_{\mathrm{arg}}(u,x) \in \mathbb{X}_{\underline{\perp}}$ means here one-step u-evaluated *argument,* and $e_{\mathrm{map}}(u,x)$ denotes the remaining part of *map code* u still to be evaluated after that evaluation step.

PR **Definition** of step e, p. r. on $\mathrm{depth}(u) \in \mathbb{N}$, now runs as follows:

- $\mathrm{depth}(u) = 0$, i. e. u of form $\ulcorner\mathrm{ba}\urcorner$,

  $$\mathrm{ba} \in \mathrm{bas}\ =_{\mathrm{by\,def}}\ \{\mathring{\mathrm{id}}, \mathring{0}, \mathring{s}, \mathring{\Pi}, \mathring{\Delta}, \mathring{\ell}, \mathring{r}\}$$

  one of the basic map constants of theory $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}$:

  $$\begin{aligned} e_{\mathrm{arg}}(\ulcorner\mathrm{ba}\urcorner, x)\ &=_{\mathrm{def}}\ \mathrm{ba}(x) \in \mathbb{X}_{\underline{\perp}}\,, \\ e_{\mathrm{map}}(\ulcorner\mathrm{ba}\urcorner, x)\ &=_{\mathrm{def}}\ \ulcorner\mathrm{id}\urcorner \in \mathrm{PR}\mathbb{X}. \end{aligned}$$

- cases of internal composition:

  $$\begin{aligned} &e\,(\langle v \odot \ulcorner\mathrm{ba}\urcorner\rangle, x)\ =_{\mathrm{def}}\ (v, \mathrm{ba}(x)) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \\ &\quad \text{and for } u \notin \{\ulcorner\mathrm{ba}\urcorner : \mathrm{ba} \in \mathrm{bas}\} : \\ &e\,(\langle v \odot u\rangle, x)\ =_{\mathrm{def}}\ (\langle v \odot e_{\mathrm{map}}(u,x)\rangle, e_{\mathrm{arg}}(u,x)) : \end{aligned}$$

step-evaluate first map code u, on argument x, and preserve remainder of u followed by v as map code to be step-evaluated on intermediate argument $e_{\mathrm{arg}}(u, x)$.

- cartesian cases:

$$e\,(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{id} \urcorner \rangle, \langle y; z \rangle) \ =_{\mathrm{def}} \ (\ulcorner \mathrm{id} \urcorner, \langle y; z \rangle) \in \mathrm{PR}\mathbb{X} \times \mathbb{X},$$

a terminating case.

For $\langle u \# v \rangle \neq \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{id} \urcorner \rangle$:

$$e\,(\langle u \# v \rangle, \langle y; z \rangle)$$
$$=_{\mathrm{def}} \ (\langle e_{\mathrm{map}}(u, y) \# e_{\mathrm{map}}(v, z) \rangle, \langle e_{\mathrm{arg}}(u, y); e_{\mathrm{arg}}(v, z) \rangle),$$

evaluate u and v in parallel.

Here free variable x on $\mathbb{X}$ legitimatly runs only on $\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \subset \mathbb{X}$, takes there the pair form $\langle y; z \rangle$. $x \in \mathbb{X} \smallsetminus \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$ results in present evaluation case into $\underline{\perp}$.

- Cases of an induced (redundant via $\ulcorner \Delta \urcorner$ and $\odot$):

$$e\,(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle, z) \ =_{\mathrm{def}} \ (\ulcorner \mathrm{id} \urcorner, \langle z; z \rangle),$$

a terminating case.

For $\langle u; v \rangle \neq \langle \ulcorner \mathrm{id} \urcorner ; \ulcorner \mathrm{id} \urcorner \rangle$:

$$e\,(\langle u; v \rangle, z)$$
$$=_{\mathrm{def}} \ (\langle e_{\mathrm{map}}(u, z); e_{\mathrm{map}}(v, z) \rangle, \langle e_{\mathrm{arg}}(u, z); e_{\mathrm{arg}}(v, z) \rangle),$$

evaluate both components u and v.

- iteration case, with $\$:= \ulcorner\S\urcorner$ designating internal *iteration:*

  $$\begin{aligned}&e\,(u^{\$}, \langle y; \nu n\rangle) = (u^{[n]}, y) : \\ &\mathrm{PR}\mathbb{X} \times \mathbb{X} \supset \mathrm{PR}\mathbb{X} \times \langle \mathbb{X} \dot{\times} \nu\mathrm{N}\rangle \to \mathrm{PR}\mathbb{X} \times \mathbb{X}.\end{aligned}$$

  Here $\nu n \in \nu\mathrm{N}$ free, $n := \nu^{-1}(\nu n) \in \mathrm{N}$, and $u^{[n]}$ is given by *code expansion* as

  $$u^{[0]} \;=_{\mathrm{def}}\; \ulcorner\mathrm{id}\urcorner,\; u^{[n+1]} \;=_{\mathrm{def}}\; \langle u \odot u^{[n]}\rangle.$$

- trash case $e\,(u, x) = (\ulcorner\mathrm{id}\urcorner, \underline{\perp}\,) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}$ if (u, x) in none of the above—regular—cases.

For to convince ourselves on termination of iteration of step e : $\mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}$—on a pair of form $(\ulcorner\mathrm{id}\urcorner, x)$—we introduce:

(*Descending*) *complexity*

$$c_{ev}(u, x) = c\,(u) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \xrightarrow{l} \mathrm{PR}\mathbb{X} \to \mathrm{N}[\omega]$$

defined p. r. as

$$\begin{aligned}
&c\,(\ulcorner\mathrm{id}\urcorner) \;=_{\mathrm{def}}\; 0 = 0 \cdot \omega \in \mathrm{N}[\omega], \\
&c\,(\ulcorner\mathrm{ba}'\urcorner) \;=_{\mathrm{def}}\; 1 \in \mathrm{N}[\omega] \\
&\quad\text{for ba}' \text{ one of the other basic map constants in bas}, \\
&c\,\langle v \odot u\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1 = c\,(u) + c\,(v) + 1 \cdot \omega^0 \in \mathrm{N}[\omega], \\
&c\,\langle u\# v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1, \\
&c\,\langle u; v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1, \\
&c\,(u^{\$}) \;=_{\mathrm{def}}\; (c\,(u) + 1) \cdot \omega^1 \in \mathrm{N}[\omega].
\end{aligned}$$

$[\,(\_)\cdot\omega^1$ is to account for unknown *iteration count* n in argument $\langle x; n\rangle$ before code expansion.$\,]$

Example: Complexity of *addition* $+ \ =_{\text{by def}}\ s^{\S} : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$:

$$c\ulcorner+\urcorner = c\ulcorner s^{\S}\urcorner = c\,(\ulcorner s\urcorner^{\$})$$
$$= (c\ulcorner s\urcorner + 1)\cdot\omega^1 = 2\cdot\omega \in \mathbb{N}[\omega] \quad [\,\equiv\ 0;2\ \in\mathbb{N}^+\,]$$

Motivation for the above definition—in particular for this latter iteration case—will become clear with the corresponding case in proof of **descent Lemma** below for *evaluation*

$$ev = ev\,(u,v)\ =_{\text{def}}\ r\,\widehat{\circ}\,\mathrm{wh}\,[\,c_{ev} > 0\,,\ e\,] : \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp}$$

defined by a <u>while</u> loop which reads

$$\underline{\text{while}}\ c_{ev}(u) > 0\ \underline{\text{do}}\ (u,x) := e(u,x)\ \underline{\text{od}}.$$

Evaluation *step* and *complexity* above are in fact the right ones to give

Basic descent lemma: For formally *partially defined* and "nevertheless" *epi-terminating* evaluation map: the defined-arguments p. r. enumeration of partial map is epi—this by axiom scheme (π)—,

$$ev = ev\,(u,x)\ =_{\text{by def}}\ r\,\widehat{\circ}\,\mathrm{wh}\,[\,c_{ev} > 0, e\,] :$$
$$\mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp}$$
$$(\text{epi-terminating within theory } \pi\mathbf{R} = \mathbf{PRa} + (\pi))$$

i. e. for step $e = e\,(u,x) = (e_{\text{map}}, e_{\text{arg}}) : \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \to \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp}$ and complexity $c_{ev} = c_{ev}(u,x)\ =_{\text{def}}\ c\,(u) : \mathrm{PR}\mathbb{X}\to\mathbb{N}[\omega]$, we have descent *above* $0 \in \mathbb{N}[\omega]$, and Stationarity *at* complexity 0 :

$$\begin{aligned}
&\mathbf{PR}\mathbb{X} \vdash c_{ev}(u,x) > 0 \implies c_{ev}\, e\,(u,x) < c_{ev}(u,x) : \\
&\qquad \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp} \to \mathbb{N}[\omega] \times \mathbb{N}[\omega] \to \mathbb{2} \text{ i.e.} \\
&\mathbf{PR}\mathbb{X} \vdash c\,(u) > 0 \implies c\, e_{\mathrm{map}}(u,x) < c\,(u) \qquad \text{(Desc)} \\
&\qquad \text{as well as} \\
&\mathbf{PR}\mathbb{X} \vdash c\,(u) \doteq 0 \quad [\iff \quad u \equiv \ulcorner \mathrm{id} \urcorner\,] \\
&\qquad \implies c_{ev}\, e\,(u,x) \doteq 0 \ \wedge\ e\,(u,x) \doteq (u,x) \qquad \text{(Sta)}
\end{aligned}$$

This with respect to the canonical, *reverse-lexicographic,* and—intuitively—*finite-descent* order of polynomial semiring $\mathbb{N}[\omega]$.

Proof: The only non-trivial case $(v,b) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}$ for descent $c_{ev}\, e\,(v,b) < c_{ev}(v,b)$ is iteration case $(v,b) = (u^{\$}, \langle x; n \rangle)$. In this "acute" iteration case we have

$$\begin{aligned}
&c\,(u^{[n]}) = c\,(\langle u \odot \langle u \ldots \odot u \rangle \ldots \rangle) \\
&\ = n \cdot c\,(u) + (n \dot{-} 1) < \omega \cdot (c(u) + 1) = c(u^{\$}),
\end{aligned}$$

proved in detail by induction on n **q. e. d.**

5.4 Evaluation characterisation

Dominated characterisation theorem for evaluation:
$ev = ev\,(u,a) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ is characterised by

- $\mathbf{PR}\mathbb{X}\mathbf{a} \vdash\ [\, ev\,(\ulcorner \mathrm{ba} \urcorner, x) \doteq \mathrm{ba}(x)\,]$

  as well as, again within $\mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R}$ and strengthenings, by:

- $\quad [\,m \text{ } \mathit{deff} \text{ } \mathit{ev}\,(v \odot u, x)\,] \implies$
  $\mathit{ev}\,(\langle v \odot u\rangle, x) \doteq \mathit{ev}\,(v, \mathit{ev}\,(u, x));$

  this reads: if m *defines* the left hand iteration ev, i. e. if iteration ev of *step e terminates* on the left hand argument after at most m steps, then ev terminates in at most m steps on right hand side as well, and the two evaluations have equal results.

- $\quad [\,m \text{ } \mathit{deff} \text{ } \mathit{ev}\,(\langle u\#v\rangle, \langle x; y\rangle)\,] \implies$
  $\mathit{ev}\,(\langle u\#v\rangle, \langle x; y\rangle) \doteq \langle \mathit{ev}\,(u, x); \mathit{ev}\,(v, y)\rangle,$

  $\quad [\,m \text{ } \mathit{deff} \text{ } \mathit{ev}\,(\langle u; v\rangle, z)\,] \implies$
  $\mathit{ev}\,(\langle u; v\rangle, z) \doteq \langle \mathit{ev}\,(u, z); \mathit{ev}\,(v, z)\rangle.$

- $\mathit{ev}\,(u^{\$}, \langle x; \ulcorner 0 \urcorner\rangle) \doteq x,$

  $\quad [\,m \text{ } \mathit{deff} \text{ } \mathit{ev}\,(u^{\$}, \langle x; \nu(s\,n)\rangle\,] \implies:$

  $\quad [\,m \text{ } \mathit{deff} \text{ all } \mathit{ev} \text{ below}\,] \wedge$
  $\mathit{ev}\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \doteq \mathit{ev}\,(u, \mathit{ev}\,(u^{\$}, \langle x; \nu\,n\rangle)).$

- it *terminates,* with all properties above, when situated in a set theory $\mathbf{T}$, since there complexity receiving ordinal $\mathbb{N}[\omega]$ has (only) finite descent, in terms of existential quantification.

Corollary: within $\mathbf{T}$, we have the double recursive equations

- $\mathit{ev}\,(\ulcorner \mathrm{ba} \urcorner, x) \doteq \mathrm{ba}(x),$

- $\mathit{ev}\,(\langle v \odot u\rangle, x) \doteq \mathit{ev}\,(v, \mathit{ev}\,(u, x)),$

- $\mathit{ev}\,(\langle u\#v\rangle, \langle x; y\rangle) \doteq \langle \mathit{ev}\,(u, x); \mathit{ev}\,(v, y)\rangle,$
  $\mathit{ev}\,(\langle u; v\rangle, z) \doteq \langle \mathit{ev}\,(u, z); \mathit{ev}\,(v, z)\rangle,$

- $ev\,(u^{\$}, \langle x; \ulcorner 0 \urcorner \rangle) \doteq x,$ and
  $ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \doteq ev\,(u, ev\,(u^{\$}, \langle x; \nu\,n\rangle)).$

Within **T**—as well as within partial p. r. theories $\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}, \pi\widehat{\mathbf{R}}$—these equations can be taken as **definition** for $\mathbf{PR}\mathbb{X}$ code evaluation ev. Within **T**, they define evaluation as a total map.

Proof of **theorem** by primitive recursion (Peano Induction) on $m \in \mathbb{N}$ free, via case distinction on codes w, and arguments $z \in \mathbb{X}$ appearing in the different cases of the asserted conjunction (case w one of the basic map constants being trivial). All of the following—**induction step**—is situated in $\mathbf{PR}\mathbb{X}\mathbf{a}$, read: $\mathbf{PR}\mathbb{X}\mathbf{a} \vdash$ etc. If you are interested first in the negative results for set theories **T**, you can read it "$\mathbf{T} \vdash \ \ldots$" but **T** still deriving properties just of $\mathbf{PR}\mathbb{X}$ map codes.

- case $(w, z) = (\langle v \odot u\rangle, x)$ of an (internally) *composed,* subcase $u = \ulcorner \mathrm{id} \urcorner$: obvious.

  Non-trivial subcase $(w, z) = (\langle v \odot u\rangle, x)$, $u \neq \ulcorner \mathrm{id} \urcorner$:

$$
\begin{aligned}
&\quad m+1 \ \mathit{deff} \ \ ev\,(\langle v \odot u\rangle, x) \implies : \\
&ev\,(\langle v \odot u\rangle, x) \doteq e^{\S}((\langle v \odot e_{\mathrm{map}}(u, x)\rangle, e_{\mathrm{arg}}(u, x)), m) \\
&\quad \text{by iterative definition of } ev \text{ in this case} \\
&\doteq ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \\
&\quad \text{by induction hypothesis on } m \\
&\implies : \\
&m+1 \ \mathit{deff} \ \ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \\
&\wedge \ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \doteq ev\,(v, ev\,(u, x)) \ :
\end{aligned}
$$

The latter implication "holds" same way back, by the same induction hypothesis on m (map code v unchanged.)

- case $(w, z) = (\langle u\#v\rangle, \langle x; y\rangle)$ of an (internal) *cartesian product:* Obvious by definition of ev on a cartesian product map codes. Pay attention to arguments out of $\mathbb{X} \smallsetminus \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle$ evaluated into $\underline{\perp}$ in this case (and in similar cases). In more detail:

$$\begin{aligned}
&ev\,(w, z) :=\\
&ev\,(\langle u\#v\rangle, \langle x; y\rangle)\\
&\quad =_{\text{by def}}\ ev\,(\langle e_{\text{map}}(u, x)\#e_{\text{map}}(v, y)\rangle, \langle e_{\text{arg}}(u, x), e_{\text{arg}}(v, y)\rangle)\\
&\quad\ \doteq \langle ev(e_{\text{map}}(u, x), e_{\text{arg}}(u, x)), ev(e_{\text{map}}(v, y), e_{\text{arg}}(v, y))\rangle\\
&\quad\ \ \in \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle
\end{aligned}$$

- alternatively (or both): case $(w, z) = (\langle u; v\rangle, z)$ of an internal induced:

$$ev(w, z) \doteq \langle ev(u, z), ev(v, z)\rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle.$$

- case $(w, z) = (u^{\$}, \langle x; \ulcorner 0 \urcorner\rangle)$ of a null-fold (internally) iterated: again obvious.

- case $(w, z) = (u^{\$}, \langle x; \nu(s\,n)\rangle)$ of a genuine (internally) iterated:

$$
\begin{aligned}
&\quad m+1 \text{ \textit{deff} } ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \implies \\
&\quad m+1 \text{ \textit{deff} all instances of } ev \text{ below, and:} \\
&ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \\
&\doteq ev\,(e_{\mathrm{map}}(u^{\$}, \langle x; \nu(s\,n)\rangle), e_{\mathrm{arg}}(u^{\$}, \langle x; \nu(s\,n)\rangle)) \\
&\doteq ev\,(u^{[n+1]}, x) \doteq ev\,(\langle u \odot u^{[n]}\rangle, x) \doteq ev\,(u, ev\,(u^{[n]}, x)) \\
&\quad \text{the latter by induction hypothesis on } m, \\
&\quad \text{case of internal composed} \\
&\doteq ev\,(u, \langle ev\,(u^{\$}, x); \nu\, n\rangle) : \text{same way back.}
\end{aligned}
$$

This shows the (remaining) predicative *iteration* equations "anchor" and "step" for an (internally) iterated $u^{\$}$, and so **proves** fullfillment of the above double recursive system of equations for $ev : \mathrm{PR}\mathbb{X}\mathrm{a} \times \mathbb{X} \rightharpoonup \mathbb{X}$ subordinated to *global* evaluation $ev : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ **q. e. d.**

Characterisation corollary: Evaluation—$\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}$ map—

$$ev = ev\,(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$$

defined as *complexity controlled iteration*—CCI—with complexity values in ordinal $\mathbb{N}[\omega]$, epi-terminates in theory $\pi\widehat{\mathbf{R}}$: has epimorphic defined arguments enumeration. This by definition of this theory strengthening $\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}$. **And** it satisfies there the characteristic double-recursive equations above for evaluation ev.

Objectivity theorem: Evaluation ev is *objective,* i. e. for each *single,* (meta free) $f : A \to B$ in theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ itself, we have

$$\begin{aligned}
&\mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R} \vdash\ [\, m \ \mathit{deff}\ \ ev(\ulcorner f \urcorner, a)\,] \implies \\
&ev(\ulcorner f \urcorner, a) = f(a), \text{ symbolically:} \\
&\pi\mathbf{R} \vdash\ ev(\ulcorner f \urcorner, _\,) = f : A \rightharpoonup B.
\end{aligned}$$

For frame a set theory $\mathbf{T}$, there is no need for explicit domination m *deff* etc.

Proof by substitution of codes of $\mathbf{PR}\mathbb{X}\mathbf{a}$ maps into code variables $u, v, w \in \mathrm{PR}\mathbb{X} \subset \mathbb{N}$ in Evaluation Characterisation above, in particular:

- $[\, m \ \mathit{deff}\ \ ev\,(\ulcorner g \circ f \urcorner, a)\,] \implies$
  $ev\,(\langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle, a) \doteq ev\,(\ulcorner g \urcorner, ev\,(\ulcorner f \urcorner, a)),$
  $\doteq g(f(a)) \doteq (g \circ f)(a)$ recursively (on m) and

- $[\, m \ \mathit{deff}\ \ ev\,(\ulcorner f^{\S} \urcorner, \langle a; \nu(s\,n)\rangle\,] \implies :$

  $[\, m \ \mathit{deff}\ \text{all}\ \ ev\ \ \text{below}\,] \wedge$

  $ev\,(\ulcorner f \urcorner^{\$}, \langle a; \nu(s\,n)\rangle) \doteq ev\,(\ulcorner f \urcorner, ev\,(\ulcorner f \urcorner^{\$}, \langle a; \nu\, n\rangle))$

  $\doteq f(f^{\S}(a, \nu n)) = f^{\S}(a, \nu(s\,n))$ recursively on m.

- it *terminates,* with this objectivity, within set theory $\mathbf{T}$.

Chapter 6

PR Decidability by Set Theory

We embed evaluation $ev(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ of PR map codes into **set** theory, theory $\mathbf{T}$.

Notion $f =^{\mathbf{PR}} g$ of p. r. maps is externally p. r. enumerated, by complexity of (binary) deduction trees.

Internalising—*formalising*—gives internal notion of p. r. equality (not: stronger $\mathbf{T}$-equality)

$$u \check{=}_k v \in \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X}$$

coming by internal *deduction tree* dtree_k, which can be canonically provided with arguments in $\mathbb{X}$—top down from (suitable) argument x given to the *root* equation $u \check{=}_k v$ of dtree_k.

We denote internal deduction tree argumented this way by dtree_k/x, *root* of dtree_k/x then is $u/x \check{=}_k v/x$.

6.1 PR soundness framed by set theory

PR Evaluation *soundness* theorem Framed by set theory T : For p. r. theory **PR** with its internal notion of equality '$\check{=}$' we have:

(i) PR$\mathbb{X}$ to **T** evaluation **soundness:**

$$\mathbf{T} \vdash u \check{=}_k v \implies ev(u,x) = ev(v,x) \qquad (\bullet)$$

Substituting in the above "concrete" **PR**$\mathbb{X}$**a** codes into u resp. v, we get, by *objectivity* of evaluation ev :

(ii) **T**-Framed Objective soundness of **PR** :

For **PR**$\mathbb{X}$**a** maps $f, g : \mathbb{X} \supset A \to B \subset \mathbb{X}$:

$$\mathbf{T} \vdash \ulcorner f \urcorner \check{=} \ulcorner g \urcorner \implies f(a) = g(a).$$

(iii) Specialising to case $u := \ulcorner \chi \urcorner$, $\chi : \mathbb{X} \to \mathbb{2}$ a p. r. *predicate,* and to $v := \ulcorner \mathrm{true} \urcorner$, we get

T-framed *Logical soundness of* **PR** :

$$\mathbf{T} \vdash \ \exists\, k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall\, x\, \chi(x) :$$

If *a p. r. predicate is—within* **T**—**PR**-*internally* provable, *then it holds in* **T** *for all of its arguments.*

Proof of logically central assertion ($\bullet$) by primitive recursion on k, dtree_k the k th deduction tree of the theory. These (argument-free) deduction trees are counted in lexicographical order.

Remark: A detailed proof is given for frame theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ and termination-conditioned evaluations in next section. This proof logically includes present case of frame theory a set theory $\mathbf{T}$: within such $\mathbf{T}$ as frame, both evaluations, *ev* as well as *deduction tree evaluation* ev_d, terminate on all of their arguments.

Super Case of *equational* internal axioms:

- associativity of (internal) composition:

  $\langle\langle w \odot v\rangle \odot u\rangle \check{=}_k \langle w \odot \langle v \odot u\rangle\rangle \implies$

  $$\begin{aligned} &ev\,(\langle w \odot v\rangle \odot u, x) = ev\,(\langle w \odot v\rangle, ev\,(u, x)) \\ &= ev\,(w, ev\,(v, ev\,(u, x))) \\ &= ev\,(w, ev\,(\langle v \odot u\rangle, x)) = ev\,(w \odot \langle v \odot u\rangle, x). \end{aligned}$$

  This proves assertion (•) in present *associativity-of-composition* case.

- Analogous **proof** for the other flat, equational cases, namely *reflexivity of equality, left and right neutrality* of $\mathrm{id} \;=_{\mathrm{by\,def}}\; \mathrm{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement's equations for the induced map as well as surjective pairing and *distributivity equation for composition with an induced.*

- **proof** of (•) for the last equational case, the

  Iteration **step,** case of *genuine iteration equation*

$$\mathrm{dtree}_k \;=\; \langle u^\$ \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle \check{\doteq}_k\, u \odot u^\$ \rangle :$$

$$\begin{aligned} \mathbf{T} \vdash\; & ev\,(u^\$ \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n) \rangle) && (1) \\ & = ev\,(u^\$, ev(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n) \rangle)) \\ & = ev\,(u^\$, \langle y; \nu(s\,n) \rangle) \\ & = ev\,(u, ev(u^\$, \langle y; \nu(n) \rangle)) \\ & = ev\,(u \odot u^\$, \langle y; \nu(n) \rangle). && (2) \end{aligned}$$

Proof of termination-conditioned inner soundness for the remaining *deep*—genuine Horn cases—for dtree_k, Horn type *deduction* of *root:*

Transitivity-of-equality case: with map code variables u, v, w we start here with argument-free deduction tree

$$\mathrm{dtree}_k \quad = \quad \Uparrow \frac{u \check{\doteq}_k\, w}{u \check{\doteq}_i\, v \;\; \wedge \;\; v \check{\doteq}_j\, w}$$

Evaluate at argument x and get in fact

$$\begin{aligned} & \mathbf{T} \vdash u \check{\doteq}_k\, w \\ & \qquad \Longrightarrow\; ev(u, x) = ev(v, x) \;\wedge\; ev(v, x) = ev(w, x) \\ & \qquad \text{(by hypothesis on } i, j < k) \\ & \qquad \Longrightarrow\; ev(u, x) = ev(w, x) : \end{aligned}$$

transitivity export q. e. d. in this case.

Case of symmetry axiom scheme for equality is now obvious.

Compatibility case of composition with equality

$$\text{dedu}_k \quad = \quad \Uparrow \frac{\langle v \odot u \rangle \check{=}_k \langle v \odot u' \rangle}{u \check{=}_i u'}$$

By induction hypothesis on $i < k$ we have

$$\begin{aligned}
&\langle v \odot u \rangle \check{=}_k \langle v \odot u' \rangle \implies : \\
&[ev(u, x) = ev(u', x) \implies \\
&ev(v \odot u, x) = ev(v, ev(u, x)) = ev(v, ev(u', x)) \\
&= ev(v \odot u', x)]
\end{aligned}$$

by hypothesis on $u \check{=}_i u'$ and by Leibniz' substitutivity, q. e. d. in this 1st compatibility case.

Case of composition with equality in second composition factor:

$$\text{dedu}_k \quad = \quad \Uparrow \frac{\langle v \odot u \rangle \check{=}_k \langle v' \odot u \rangle}{v \check{=}_i v'}$$

[Here dtree_i is not (yet) provided with all of its arguments, it *is* completly argumented during top down tree evaluation.]

$$\begin{aligned}
&\langle v \odot u \rangle \check{=}_k \langle v' \odot u \rangle \implies : \\
&ev(\langle v \odot u \rangle, x) = ev(v, ev(u, x)) = ev(v', ev(u, x)) \qquad (*) \\
&= ev(\langle v' \odot u \rangle, x).
\end{aligned}$$

$(*)$ holds by $v \check{=}_i v'$, induction hypothesis on $i < k$, and Leibniz' substitutivity: same argument into equal maps.

This proves soundness assertion ($\bullet$) in this 2nd compatibility case.

(Redundant) Case of compatibility of forming the induced map, with equality is analogous to compatibilities above, even easier, since the two map codes concerned are independent from each other.

(Final) Case of Freyd's (internal) uniqueness of the *initialised iterated,* is case

$$\mathrm{dedu}_k/\langle y;\nu(n)\rangle$$
$$= \frac{w/\langle y;\nu(n)\rangle \check{\doteq}_k \langle v^{\$} \odot \langle u\#\ulcorner\mathrm{id}\urcorner\rangle/\langle y;\nu(n)\rangle\rangle}{\mathrm{root}\,(t_i) \qquad\qquad\qquad\qquad \mathrm{root}\,(t_j)}$$

where

$$\begin{aligned}
&\mathrm{root}\,(t_i)\\
&= \langle w \odot \langle\ulcorner\mathrm{id}\urcorner;\ulcorner 0\urcorner \odot \ulcorner\Pi\urcorner\rangle/y \check{\doteq}_i u/y\rangle,\\
&\mathrm{root}\,(t_j)\\
&= \langle w \odot \langle\ulcorner\mathrm{id}\urcorner\#\ulcorner s\urcorner\rangle/\langle y;\nu(n)\rangle \check{\doteq}_j \langle v \odot w\rangle/\langle y;\nu(n)\rangle\rangle.
\end{aligned}$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal p. r. equations as $\langle v^{\$} \odot \langle u\#\ulcorner\mathrm{id}\urcorner\rangle\rangle$. It should be—**is**: *soundness*—evaluated equal to the latter, on $\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \subset \mathbb{X}$.

Soundness assertion ($\bullet$) for the present Freyd's *uniqueness* case recurs on $\check{\doteq}_i$, $\check{\doteq}_j$ turned into predicative equations '$=$', these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of '$=$' and established properties of basic evaluation *ev* of map terms.

So here is the remaining—inductive—**proof,** prepared by

$$\mathbf{T} \vdash ev\,(w, \langle y; \nu(0)\rangle) = ev\,(u; y) \qquad (\bar{0})$$

as well as

$$\begin{aligned} ev(w, \langle y; \nu(s\,n)\rangle) &= ev\,(w, \langle y; \ulcorner s \urcorner \odot \nu(n)\rangle) \\ &= ev\,(w \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner\rangle, \langle y; \nu(n)\rangle) \\ &= ev\,(v \odot w, \langle y; \nu(n)\rangle), \qquad (\bar{s}) \end{aligned}$$

the same being true for $w' := v^{\$} \odot \langle u \# \ulcorner \mathrm{id} \urcorner\rangle$ in place of w, once more by (characteristic) double recursive equations for ev, this time with respect to the *initialised internal iterated* itself.

$(\bar{0})$ and $(\bar{s})$ put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$—all other free variables k, u, v, w, y together form the *passive parameter* for this induction—*truncated soundness* assertion $(\bullet)$ for this *Freyd's uniqueness* case, namely

$$\mathbf{T} \vdash ev\,(w, \langle y; \nu(n)\rangle) = ev\,(v^{\$} \odot \langle u \# \ulcorner \mathrm{id} \urcorner\rangle, \langle y; \nu(n)\rangle).$$

Induction runs as follows:

Anchor $n = 0$:

$ev\,(w, \langle y; \nu(0)\rangle) = ev\,(u, y) = ev\,(w', \langle y; \nu(0)\rangle),$

step:

$$\begin{aligned} &ev\,(w, \langle y; \nu(n)\rangle) = ev\,(w', \langle y; \nu(n)\rangle) \implies : \\ &ev\,(w, \langle y; \nu(s\,n)\rangle) = ev\,(v, ev\,(w, \langle y; \nu(n)\rangle)) \\ &= ev\,(v, ev\,(w', \langle y; \nu(n)\rangle)) = ev\,(w', \langle y; \nu(s\,n)\rangle), \end{aligned}$$

the latter since evaluation ev preserves predicative equality '$=$' (Leibniz) **q. e. d.**

6.2 PR-predicate decision by set theory

We consider here $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicates for decidability by set theorie(s) $\mathbf{T}$. Basic tool is **T-framed soundness of $\mathbf{PR}\mathbb{X}\mathbf{a}$** just above, namely

$$\frac{\chi = \chi(a) : A \to \mathbb{2} \ \ \mathbf{PR}\mathbb{X}\mathbf{a} \text{ predicate}}{\mathbf{T} \vdash \ \exists k \, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner \chi \urcorner) \implies \forall a \, \chi(a).}$$

Within **T define** for $\chi : A \to \mathbb{2}$ out of $\mathbf{PR}\mathbb{X}\mathbf{a}$ a partially defined (alleged, individual) μ-recursive decision $\nabla\chi = \nabla_{\mathrm{PR}}\chi : \mathbb{1} \rightharpoonup \mathbb{2}$ by first fixing *decision domain*

$$D = D\chi := \{k \in \mathbb{N} : \neg\, \chi(\mathrm{ct}_A(k)) \ \vee \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner \chi \urcorner)\},$$

$\mathrm{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count of A; and then, with (partial) recursive $\mu D : \mathbb{1} \rightharpoonup D \subseteq \mathbb{N}$ within $\mathbf{T}$:

$$\nabla\chi \ =_{\mathrm{def}} \begin{cases} \text{false if } \neg\, \chi(\mathrm{ct}_A(\mu D)) \\ \quad (\textit{counterexample}), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(\mu D, \ulcorner \chi \urcorner) \\ \quad (\textit{internal proof}), \\ \perp \ (\textit{undefined}) \text{ otherwise, i.e.} \\ \quad \text{if } \forall a \, \chi(a) \ \wedge \ \forall k \, \neg\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner \chi \urcorner). \end{cases}$$

[This (alleged) decision is apparently (μ-)recursive within $\mathbf{T}$, even if apriori only partially defined.]

There is a first *consistency* problem with this definition: are the *defined* cases *disjoint?*

Yes, within frame theory $\mathbf{T}$ which *soundly frames* theory $\mathbf{PR}\mathbb{X}\mathbf{a}$:

$$\mathbf{T} \vdash \ (\exists\, k \in \mathbb{N})\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner) \implies \forall a\,\chi(a).$$

$\mathbf{T}$-framed $\mathbf{PR}\mathbb{X}\mathbf{a}$-soundness leads to

Complete T derivation alternative for $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate χ :

(a) $\mathbf{T} \vdash \ \nabla\chi = \mathrm{false}$ iff $\mathbf{T} \vdash \ \exists a \neg\,\chi(a)$,

(b) $\mathbf{T} \vdash \ \nabla\chi = \mathrm{true}$ iff $\mathbf{T} \vdash \ \exists k\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$

iff $\mathbf{T} \vdash \ \exists k\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner) \wedge \forall a\,\chi(a)$,

the latter iff by $\mathbf{T}$-*framed soundness* of $\mathbf{PR}\mathbb{X}\mathbf{a}$.

(c) $\mathbf{T} \vdash \ \nabla\chi = \bot$ iff $\mathbf{T} \vdash \ \forall a\,\chi(a) \wedge \forall k \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$.

Remark:

- within quantified arithmetic $\mathbf{T}$ we have the right to replace $\chi(\mathrm{ct}_A(\mu D))$ by $\exists\, a\,(\chi(a))$ in the above, and $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(\mu D, \ulcorner\chi\urcorner)$ by $\exists\, k\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$.

- for consistent $\mathbf{T}$, χ an arbitrary $\mathbf{T}$-formula, and *Proof* $\mathrm{Prov}_{\mathbf{T}}$ in place of $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}$, *soundness*—and therefore *disjointness* of (termination) cases(a) and (b) above—does not work anymore: take for χ Gödel's undecidable formula φ with its "characteristic" property

$$\mathbf{T} \vdash \ \neg\,\varphi \iff \exists k\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner).$$

Merging now the (right hand sides) of the latter two cases gives the following complete alternative,

Decidability of primitive recursive free-variable predicates *by* quantified extension $\mathbf{T}$ (via μ-recursive decision algorithm $\nabla\chi : \mathbb{1} \rightharpoonup \mathbb{2}$):

For (arbitrary) $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$\begin{aligned} &\mathbf{T} \vdash\ \forall a\, \chi(a) \quad \textit{or} \\ &\mathbf{T} \vdash\ \exists a \neg\, \chi(a). \end{aligned}$$

Theorem or derivable existence of a counterexample **q. e. d.**

Decision Remark: this does not mean a priori that *decision algorithm* $\nabla\chi$ terminates for all such predicates χ. The theorem says only that χ is ***decidable*** "by", *within* theory $\mathbf{T}$, that it is *not independent* from $\mathbf{T}$.

6.3 Gödel's incompleteness theorems

We visit §2. Gödel's theorems, in Smorynski 1977.

First Incompleteness Theorem. *Let* $\mathbf{T}$ *be a formal theory containing arithmetic. Then there is a sentence* φ *which asserts its own unprovability and such that:*

(i) *If* $\mathbf{T}$ *is consistent,* $\mathbf{T} \nvdash \varphi$.

(ii) *If* $\mathbf{T}$ *is* ω*-consistent,* $\mathbf{T} \nvdash \neg\, \varphi$.

In §3.2.6 Smorynski discusses possible choices of *arithmetic* (theory) $\mathbf{S}$, namely

(a) **PRA** = (classical, free-variables) primitive recursive arithmetic,

S. Feferman: "my **PRA**", in contrast to **PRa** above.

(b) **PA** = Peano Arithmetic.

Conjecture: $\mathbf{PA} \cong \mathbf{PR}\exists \sqsubset \mathbf{PRa}\exists$.

(c) **ZF** = Zermelo-Fraenkel set theory. "This is both a good and a bad example. It is bad because the whole encoding problem is more easily solved in a set theory than in an arithmetical theory. By the same token, it is a good example."

Conjecture: PRA can categorically be viewed as cartesian theory with weak NNO in Lambek's sense.

We take $\mathbf{S} := \mathbf{PRa}$, embedding extension of categorical theory **PR**, formally stronger than **PRA** because of uniqueness of maps defined by the full schema of primitive recursion, and weaker than $\mathbf{PA} \cong \mathbf{PR}\exists$.

By construction of arithmetic **PRa**, *one can adequatly encode syntax in this* $\mathbf{S} = \mathbf{PRa}$, since Smorynski's conditions (i)-(iii) for the representation of p. r. functions are fulfilled.

We take for formal extension **T** of **S** one of the categorical pendants to suitable set theories (subsystems of **ZF**, see Osius 1974), or the *(first order) elementary theory of two-valued Topoi with NNO,* cf. Freyd 1972, or, *minimal choice,* $\mathbf{T} := \mathbf{PRa}\exists \sqsupset \mathbf{PA}$.

Derivability theorem: Our **S** encoding, extended from **PRa** to **T**, meets the following (quantifier free categorically expressed) *Deriv-*

ability Conditions in §2.1 of Smorynski:

D1 $\mathbf{T} \overset{\underline{k}}{\vdash} \varphi$ infers $\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{T}}(\mathrm{num}(\underline{k}), \ulcorner\varphi\urcorner)$.

D2 $\mathbf{S} \vdash \ \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner) \implies \mathrm{Prov}_{\mathbf{T}}(j_2(k), \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner))$,
$j_2 = j_2(k) : \mathbb{N} \to \mathbb{N}$ suitable.

D3 $\mathbf{S} \vdash \ \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner) \wedge \mathrm{Prov}_{\mathbf{T}}(k', \ulcorner\varphi \Rightarrow \psi\urcorner)$
$\implies \mathrm{Prov}_{\mathbf{T}}(j_3(k, k'), \ulcorner\psi\urcorner)$,
$j_3 = j_3(k, k') : \mathbb{N}^2 \to \mathbb{N}$ suitable.

Smorynski's **proof** gives the *First Gödel's incompleteness theorem,* and from that the

Second incompleteness theorem: Let $\mathbf{T}$ be one of the extensions above of $\mathbf{PR}\exists$, and $\mathbf{T}$ consistent. *Then*

$$\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}},$$

where $\mathrm{Con}_{\mathbf{T}} = \forall k \ \neg\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner)$ *is the sentence asserting the consistency of* $\mathbf{T}$.

From this Gödel's theorem and our *PR Decidability theorem* for quantified arithmetic $\mathbf{PRa}\exists, \mathbf{T}$ we get

Inconsistency provability theorem for quantified arithmetical (set) theories $\mathbf{T}$:

If $\mathbf{T}$ is consistent, then

$$\mathbf{T} \vdash \neg\, \mathrm{Con}_{\mathbf{T}}.$$

[If not, then it derives everything, in particular $\neg\, \mathrm{Con}_{\mathbf{T}}$. We will see that p. r. arithmetic, under a mild termination condition for external evaluation, yields inconsistency of $\mathbf{T}$.]

Chapter 7

Consistency Decision within $\pi\mathbf{R}$

This final chapter is better read in overview than explained.

7.1 Termination conditioned evaluation soundness

ES[1] Theorem on termination-conditioned soundness:

For p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ [2] and internal notion of equality $\check{\doteq} = \check{\doteq}_k : \mathbb{N} \to \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X}$, dtree_k the k th deduction tree of universe theory $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$, we have:

(i) *Termination-Conditioned **Inner** soundness:*

[1] *Evaluation soundness*

[2] presumably *not* directly for $\pi\mathbf{R}$ with respect to its own internal equality, without assumption of "π-consistency," in this regard RCF 2 contains an error

With $r = r(u, x) = x : \mathrm{PR}\mathbb{X} \times \mathbb{X} \to \mathbb{X}$ right projection:

$$\begin{aligned}\mathbf{PR}\mathbb{X}\mathbf{a} \vdash \langle u \check{\doteq}_k v\rangle &\doteq \mathrm{root}\ (\mathrm{dtree}_k) \\ &\wedge\ m\ \mathit{deff}\ \ ev_d\,(\mathrm{dtree}_k/x) \\ &\implies ev\,(u, x) \doteq ev\,(v, x)\,. \qquad (\bullet)\end{aligned}$$

explicitly:

$$\begin{aligned}\mathbf{PR}\mathbb{X}\mathbf{a} \vdash u \check{\doteq}_k v\ \wedge\ c_d\, e_d^m\,(\mathrm{dtree}_k/x) \doteq 0 \\ \implies ev\,(u, x) \doteq e^m(u, x) \doteq e^m(v, x) \\ \doteq ev\,(v, x), \qquad (\bullet)\end{aligned}$$

free map-code variables u, v, variable x free in universal set $\mathbb{X}$.

[*Argumentation* dtree_k/x of dtree_k and definition of *argumented tree evaluation* ev_d based on its evaluation step e_d and complexity c_d is by merged recursion on $\mathrm{depth}(\mathrm{dtree}_k)$, within **proof** below]

In words, this "m-Truncated", "m-Dominated" Inner soundness says that theory **PRa** derives:

If *for an internal* $\mathbf{PR}\mathbb{X}$ equation $u \check{\doteq}_k v$ argumented deduction tree dtree_k/x *for* $u \check{\doteq}_k v$, argumented *with* $x \in \mathbb{X}$, *admits complete* argumented-tree evaluation, i. e.

if *tree-evaluation becomes completed after a finite number m of evaluation steps,*

then *both sides of this* internal (!) equation *are completly evaluated on x by (at most) m steps e of* basic *evaluation ev, into equal values.*

Substituting in the above "concrete" codes into u resp. v , we get, by *objectivity* of evaluation ev, formally "mutatis mutandis":

(ii) *Termination-Conditioned Objective soundness for Map Equality:*

For $\mathbf{PR}\mathbb{X}\mathbf{a}$ maps $f, g : A \to B$:

$$\begin{aligned}\mathbf{PR}\mathbb{X}\mathbf{a} \vdash [\ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \ \wedge \ m \ \mathit{deff} \ \ ev_d(\mathrm{dtree}_k/a)\,] \\ \implies f(a) \doteq_B r\ e^m(\ulcorner g \urcorner, a) \doteq_B g(a), \ a \in A \text{ free :}\end{aligned}$$

If *an internal p. r. deduction-tree for (internal) equality of* $\ulcorner f \urcorner$ *and* $\ulcorner g \urcorner$ *is available, and* ***if*** *on this tree—top down argumented with* a *in* A*—tree evaluation* ***terminates, then*** *equality* $f(a) \doteq_B g(a)$ *of* f *and* g *at this argument is the consequence.*

(iii) Specialising this to case of $f := \chi : A \to \mathbb{2}$ a p. r. *predicate* and to $g := \mathrm{true}_A : A \to \mathbb{2}$ we eventually get

Termination-Conditioned Objective Logical soundness:

$$\mathbf{PR}\mathbb{X}\mathbf{a} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \chi \urcorner) \ \wedge \ m \ \mathit{deff} \ \ ev_d(\mathrm{dtree}_k/a) \implies \chi(a) :$$

If *tree-evaluation of an internal deduction tree for a free variable p. r. predicate* $\chi : A \to \mathbb{2}$*—the tree* argumented *with* $a \in A$*—terminates after a finite number* m *of evaluation steps,* ***then*** $\chi(a) \doteq \mathrm{true}$ *is the consequence,* within $\mathbf{PR}\mathbb{X}\mathbf{a}$ as well as within its extensions $\pi\mathbf{R}$—and set theory $\mathbf{T}$.

Remark to proof below: in present case of frame theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ (and stronger theory $\pi\mathbf{R}$) we have to *control* all evaluation step iterations, and we do that by control of iterative evaluation ev_d of whole argumented deduction trees, whose recursive definition will be—merged—part of this proof.

Proof of—basic—*termination-conditioned* ***inner*** *soundness,* i. e. of implication $(\bullet)$ in *ES theorem* is by induction on deduction tree counting index $k \in \mathbb{N}$ counting family $\mathrm{dtree}_k : \mathbb{N} \to \mathrm{Bintree}$, starting with (flat) $\mathrm{dtree}_0 = \langle \ulcorner \mathrm{id} \urcorner \check{=}_0 \ulcorner \mathrm{id} \urcorner \rangle$. $m \in \mathbb{N}$ is to dominate argumented-deduction-tree evaluation ev_d to be recursively defined below: *condition*

m *deff* $ev_d(\mathrm{dtree}_k/x)$, step e_d, complexity c_d.

We argue by *recursive case distinction* on the form of the top up-to-two layers—top (implicational) deduction—dedu_k/x of argumented deduction tree dtree_k/x at hand.

Flat **super case** $\mathrm{depth}(dtree_k) = 0$, i. e. super case of *unconditioned,* axiomatic (internal) *equation* $u \check{=}_k v$:

The first involved of these cases is *associativity* of (internal) *composition:*

$$\mathrm{dtree}_k \;=\; \langle \langle w \odot v \rangle \odot u \rangle \check{=}_k \langle w \odot \langle v \odot u \rangle \rangle$$

In this case—no need of a recursion on k—

$$\begin{aligned}
\mathbf{PR}\mathbb{X}\mathbf{a} \vdash m\ deff\ ev_d(\mathrm{dtree}_k/x) \implies \\
& [\, m\ deff\ ev\,(\langle w \odot v \rangle \odot u, x)\,] \\
& \wedge [\, m\ deff\ ev(\langle w \odot v \rangle, ev\,(u, x)) \\
& \wedge [\, m\ deff\ ev(w, ev(v, ev(u, x))) \\
& \wedge [\, m\ deff\ ev(w, ev(\langle v \odot u \rangle, x)) \\
& \wedge [\, m\ deff\ ev\,(\langle w \odot \langle v \odot u \rangle \rangle, x)\,] \wedge
\end{aligned}$$

$$ev\,(\langle w \odot v\rangle \odot u, x) \doteq ev\,(\langle w \odot v\rangle, ev\,(u,x))$$
$$\doteq ev\,(w, ev\,(v, ev\,(u,x)))$$
$$\doteq ev\,(w, ev\,(\langle v \odot u\rangle, x)) \doteq ev\,(w \odot \langle v \odot u\rangle, x).$$

This proves assertion (•) in present *associativity-of-composition* case. [New in comparison to previous *Inconsistency* chapter is here only the "preamble" *m deff* etc.]

Analogous **proof** for the other flat, equational cases, namely *reflexivity of equality, left and right neutrality* of id $=_{\text{by def}}$ $\mathrm{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement's equations for the induced map as well as surjective pairing and distributivity of composition over forming the induced map.

Godement's equations $l \circ (f,g) = f,\ r \circ (f,g) = g$:

$$m \textit{ deff } ev \text{ etc.} \implies$$
$$ev(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z) \doteq r\, e^m(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z)$$
$$\doteq \mathring{\ell}(\langle ev(u,z); ev(v,z)\rangle) \doteq ev(u,z),$$

analogously for composition with right projection.

Fourman's equation $(l \circ h, r \circ h) = h$:

$$m \textit{ deff } ev \text{ etc.} \implies$$
$$ev(\langle \ulcorner \mathring{\ell} \urcorner \odot w; \ulcorner \mathring{r} \urcorner \odot w\rangle, z)$$
$$\doteq \langle ev(\ulcorner \mathring{\ell} \urcorner, ev(w,z)); ev(\ulcorner \mathring{r} \urcorner, ev(w,z))\rangle$$
$$\doteq \langle \mathring{\ell}(ev(w,z)); \mathring{r}(ev(w,z))\rangle \doteq ev(w,z)$$

by SP equation on objective level.

Now here are the **proofs**—with preambles—of (•), for the last equational case, the

Iteration step, case of *genuine iteration equation*
$\mathrm{dtree}_k \;=\; \langle u^\$ \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle \check{\doteq}_k\, u \odot u^\$ \rangle$:

$$
\begin{aligned}
\mathbf{PR}\mathbb{X}\mathbf{a} \vdash {} & m \ \mathit{deff}\ \ ev_d(\mathrm{dtree}_k/\langle y; \nu(n)\rangle) \implies \\
& m \ \mathit{deff}\ \ \textit{all instances of ev below, and:} \\
& ev\,(u^\$ \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle) && (1) \\
& \doteq ev\,(u^\$, ev(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u^\$, \langle y; \nu(s\,n)\rangle) \\
& \doteq ev\,(u^{[s\,n]}, y) \quad \text{(by definition of } ev \text{ step } e) \\
& \doteq ev\,(u \odot u^{[n]}, y) \\
& \doteq ev\,(u, ev(u^\$, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u \odot u^\$, \langle y; \nu(n)\rangle). && (2)
\end{aligned}
$$

Proof of termination-conditioned inner soundness for the remaining *deep*—genuine HORN cases—for dtree_k , HORN type (at least) at *deduction* of *root:*

Transitivity-of-equality case: with map code variables u, v, w we start here with argument-free deduction tree

$$
\mathrm{dtree}_k \quad = \quad \cfrac{\cfrac{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ji}}{u \,\check{\doteq}_i\, v} \qquad \cfrac{\mathrm{dtree}_{ij} \quad \mathrm{dtree}_{jj}}{v \,\check{\doteq}_j\, w}}{u \,\check{\doteq}_k\, w}
$$

It is argumented with argument x say, recursively spread down:

$$\mathrm{dtree}_k/x \quad = \quad \frac{\dfrac{u/x \quad v/x}{\mathrm{dtree}_{ii}/x_{ii} \quad \mathrm{dtree}_{ji}/x_{ji}} \qquad \dfrac{v/x \quad w/x}{\mathrm{dtree}_{ij}/x_{ij} \quad \mathrm{dtree}_{jj}/x_{jj}}}{u/x \quad w/x}$$

Spreading down arguments from upper level down to 2nd level must/is given explicitly, further arguments spread down is then recursive by the type of deduction (sub)trees $\mathrm{dtree}_i, \mathrm{dtree}_j,\ i, j < k$.

Now by induction hypothesis on i, j we have for tree evaluation ev_d :

$$\begin{aligned}
&u \mathrel{\check{\doteq}}_k w \wedge m \ \mathit{deff}\ \ ev_d(\mathrm{dtree}_k/x) \\
&\ \Longrightarrow\ m \ \mathit{deff}\ \ ev_d(\mathrm{dtree}_i/x), ev_d(\mathrm{dtree}_j/x) \wedge \\
&ev_d(\mathrm{dtree}_i/x) \doteq \langle \ulcorner \mathrm{id} \urcorner / ev(u, x) \doteq \ulcorner \mathrm{id} \urcorner / ev(v, x) \rangle \\
&\wedge\ \ ev_d(\mathrm{dtree}_j/x) \doteq \langle \ulcorner \mathrm{id} \urcorner / ev(v, x) \doteq \ulcorner \mathrm{id} \urcorner / ev(w, x) \rangle \\
&\ \Longrightarrow\ \ ev(u, x) \doteq ev(v, x) \ \wedge\ ev(v, x) \doteq ev(w, x) \\
&\ \Longrightarrow\ \ ev(u, x) \doteq ev(w, x).
\end{aligned}$$

and this is what we wanted to show in present transitivity of equality case.

[Transitivity axiom for equality is a main reason for necessity to consider (argumented) deduction trees: intermediate map code equalities '$\check{\doteq}$' in a transitivity chain must be each evaluated, and pertaining deduction trees may be of arbitrary high evaluation complexity]

Case of symmetry axiom scheme for equality is now obvious.

Compatibility Case of composition with equality

$$\mathrm{dtree}_k/x \quad = \quad \cfrac{\langle v \odot u\rangle/x \,\check{\doteq}_k\, \langle v \odot u'\rangle/x}{\cfrac{u/x \,\check{\doteq}_j\, u'/x}{\mathrm{dtree}_{ij}/x \quad \mathrm{dtree}_{jj}/x}}$$

By induction hypothesis on $j < k$

$$\begin{aligned}
&m \textit{ deff } ev_d(\mathrm{dtree}_k/x) \implies \\
&m \textit{ deff } ev_d(\mathrm{dtree}_j/x) \implies \\
&ev(u, x) \doteq ev(u', x) \implies \\
&ev(v \odot u, x) \doteq ev(v, ev(u, x)) \doteq ev(v, ev(u', x)) \\
&\doteq ev(v \odot u', x)
\end{aligned}$$

by dominated characterisic equations for ev and Leibniz' substitutivity, q. e. d. in this 1st compatibility case.

Spread down arguments is more involved in

Case of composition with equality in second composition factor: argument spread down merged with tree evaluation ev_d and proof of result.

$$\mathrm{dtree}_k/x \quad = \quad \begin{array}{c} \langle v \odot u\rangle/x \qquad \langle v' \odot u\rangle/x \\ \hline \\ v \mathrel{\check{\doteq}_i} v' \\ \hline \\ \mathrm{dtree}_{ii} \qquad \mathrm{dtree}_{ji} \end{array}$$

[Here dtree_i is not (yet) provided with argument, it *is* argumented during top down tree evaluation below]

m *deff* $ev_d(\mathrm{dtree}_k/x) \implies$
m *deff* all instances of ev below, and:
$ev(\langle v \odot u\rangle, x) \doteq ev(v, ev(u,x)) \doteq ev(v', ev(u,x)) \qquad (*)$
$\doteq ev(\langle v' \odot u\rangle, x).$

$(*)$ holds by Leibniz' substitutivity and

m *deff* $ev_d(\mathrm{dtree}_k/x) \implies$
m *deff* $ev_d(\mathrm{dtree}_i/ev(u,x))$
[*argumentation of* dtree_i *with*
$ev(u,x)$*—calculated en cours de route*,
extra definition of e_d]
$\implies$
m *deff* $ev(v, ev(u,x)) \doteq ev(v', ev(u,x)),$

by induction hypothesis on $i < k$: The hypothesis is independent of substituted argument, provided—and this is here the case—that dtree_i is evaluated on that argument, in $m' < m$ steps, m' suitable (minimal).

This proves assertion (•) in this 2nd compatibility case.

(Redundant) case of compatibility of forming the induced map with map equality is analogous to compatibilities above, even easier, because of almost independence of any two inducing map codes from each other.

(Final) case of Freyd's (internal) uniqueness of the *initialised iterated,* is case

$$\mathrm{dedu}_k/\langle y;\nu(n)\rangle \quad = \quad \frac{w/\langle y;\nu(n)\rangle \,\check{\doteq}_k\, \langle v^{\$} \odot \langle u\#\ulcorner\mathrm{id}\urcorner\rangle/\langle y;\nu(n)\rangle\rangle}{\mathrm{root}\,(t_i) \qquad\qquad\qquad\qquad \mathrm{root}\,(t_j)}$$

where

$$\mathrm{root}\,(t_i) = \langle w \odot \langle\ulcorner\mathrm{id}\urcorner;\ulcorner 0\urcorner \odot \ulcorner\Pi\urcorner\rangle/y \,\check{\doteq}_i\, u/y\rangle,$$
$$\mathrm{root}\,(t_j) = \langle w \odot \langle\ulcorner\mathrm{id}\urcorner\#\ulcorner s\urcorner\rangle/\langle y;\nu(n)\rangle \,\check{\doteq}_j\, \langle v \odot w\rangle/\langle y;\nu(n)\rangle\rangle$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal p. r. equations as $\langle v^{\$} \odot \langle u\#\ulcorner\mathrm{id}\urcorner\rangle\rangle$. It should be—**is**: *soundness*—evaluated equal to the latter, on $\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \subset \mathbb{X}$.

The soundness assertion (•) for the present Freyd's *uniqueness* case recurs on $\check{\doteq}_i$, $\check{\doteq}_j$ turned into predicative equations '$\doteq$', these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of '$\doteq$' and established properties of basic evaluation *ev* of map terms.

So here is the remaining—inductive—**proof,** prepared by

$$\begin{aligned}
&\mathbf{T} \vdash m \ \mathit{deff}\ \mathrm{dtree}_k/\langle y; \nu(n)\rangle \implies \\
&\quad m\ \mathit{deff}\ \text{all of the following } \mathit{ev}\text{-terms and} \\
&\quad \mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u; y) \qquad (\bar{0}) \\
&\qquad \text{as well as} \\
&\quad m\ \mathit{deff}\ \text{both of the following } \mathit{ev}\text{-terms, and} \\
&\quad \mathit{ev}(w, \langle y; \nu(s\,n)\rangle) \doteq \mathit{ev}\,(w, \langle y; \ulcorner s \urcorner \odot \nu(n)\rangle) \\
&\quad \doteq \mathit{ev}\,(w \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner\rangle, \langle y; \nu(n)\rangle) \\
&\quad \doteq \mathit{ev}\,(v \odot w, \langle y; \nu(n)\rangle), \qquad (\bar{s})
\end{aligned}$$

the same being true for $w' := v^{\$} \odot \langle u \# \ulcorner \mathrm{id} \urcorner \rangle$ in place of w, once more by (characteristic) double recursive equations for ev, this time with respect to the *initialised internal iterated* itself.

$(\bar{0})$ and $(\bar{s})$ put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$—all other free variables k, u, v, w, y together form the *passive parameter* for this induction—*truncated soundness* assertion $(\bullet)$ for this *Freyd's uniqueness* case, namely

$$\begin{aligned}
&\mathbf{T} \vdash m \ \mathit{deff}\ \mathrm{dtree}_k/\langle y; \nu(n)\rangle \implies \\
&\quad m\ \mathit{deff}\ \textit{all of the ev-terms concerned above, and} \\
&\quad \mathit{ev}\,(w, \langle y; \nu(n)\rangle) \doteq \mathit{ev}\,(v^{\$} \odot \langle u \# \ulcorner \mathrm{id} \urcorner\rangle, \langle y; \nu(n)\rangle).
\end{aligned}$$

Induction runs as follows:

Anchor $n = 0$:

$\mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u, y) \doteq \mathit{ev}\,(w', \langle y; \nu(0)\rangle),$

Step: *m deff* etc. $\implies$

$$\begin{aligned} & ev\,(w, \langle y; \nu(n)\rangle) \doteq ev\,(w', \langle y; \nu(n)\rangle) \implies : \\ & ev\,(w, \langle y; \nu(s\,n)\rangle) \doteq ev\,(v, ev\,(w, \langle y; \nu(n)\rangle)) \\ & \doteq ev\,(v, ev\,(w', \langle y; \nu(n)\rangle)) \doteq ev\,(w', \langle y; \nu(s\,n)\rangle), \end{aligned}$$

the latter since evaluation *ev* preserves predicative equality '$\doteq$' (Leibniz) **q. e. d.** *Termination Conditioned p. r. soundness theorem.*

Comment: Already for stating the evaluations, we needed the—categorical, free-variables theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}, \mathbf{PR}\mathbb{X}\mathbf{a}$ of primitive recursion, as well as—for termination, even in classial frame $\mathbf{T}$—PR complexities within $\mathbb{N}[\omega]$. Since this type of soundness is a corner stone in our approach, the above complicated categorical combinatorics seem to be necessary for the constructive framework of descent theory $\pi\mathbf{R}$.

7.2 Framed consistency

From termination-conditioned soundness—resp. from $\mathbf{T}$-framed PR soundness—we get

$\pi\mathbf{R}$-framed internal p. r. consistency corollary: For *descent* theory $\pi\mathbf{R} = \mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$, axiom (π) stating non-infinite iterative descent in *ordinal* $\mathbb{N}[\omega]$, we have

$$\begin{aligned} & \pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}}, \text{ i. e. "necessarily" in } \textit{free-variables} \text{ form:} \\ & \pi\mathbf{R} \vdash \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to 2,\ k \in \mathbb{N} \text{ free}, \\ & \quad \mathbf{T} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}} : \end{aligned}$$

theory $\pi\mathbf{R}$*—as well as set theories* $\mathbf{T}$ *as an extension of* $\pi\mathbf{R}$—derive *that no* $k \in \mathbb{N}$ *is the internal* $\mathbf{PR}\mathbb{X}$-Proof *for* $\ulcorner\text{false}\urcorner$.

Proof for this **corollary** from *termination-conditioned soundness:* By assertion (iii) of that **theorem,** with $\chi = \chi(a) := \text{false}(a) = \text{false} : \mathbb{1} \to \mathbb{2}$, we get:

Evaluation-effective internal inconsistency of $\mathbf{PR}\mathbb{X}$—i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner\text{false}\urcorner$—*implies* false :

$$\begin{aligned} &\mathbf{PR}\mathbb{X}\mathbf{a},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner)\ \wedge\ c_d\ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) \doteq 0 \\ &\qquad\qquad \implies \text{false}. \end{aligned}$$

Contraposition to this, still with $k, m \in \mathbb{N}$ free:

$$\pi\mathbf{R} \vdash\ \text{true} \implies \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) \vee c_d\ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) > 0,$$

i. e. by free-variables (boolean) tautology:

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) \implies c_d\ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) > 0 : \mathbb{N}^2 \to \mathbb{2}.$$

For k "fixed", the conclusion of this implication—m free—means infinite descent in $\mathbb{N}[\omega]$ of iterative argumented deduction-tree evaluation ev_d on $\mathrm{dtree}_k/0$, which is excluded intuitively. Formally it is excluded within our theory $\pi\mathbf{R}$ taken as frame:

We apply non-infinite-descent scheme (π) to ev_d, which is given by *step* e_d and complexity c_d—the latter descends (this is *argumented-tree evaluation descent*) with each application of e_d, as long as complexity $0 \in \mathbb{N}[\omega]$ is not ("yet") reached. We combine this with—choice of—*overall "negative"* condition

$$\psi = \psi(k) := \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) : \mathbb{N} \to \mathbb{2},\ \ k \in \mathbb{N} \text{ free}$$

and get—by that scheme (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free, i. e.}$$
$$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}} \quad \textbf{q. e. d.}$$

So "slightly" strengthened theory $\pi\mathbf{R} = \mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$ derives free variables Consistency Formula for theory $\mathbf{PR}\mathbb{X}$ of primitive recursion.

Scheme (π) holds in **set** theory, since there $O := \mathbb{N}[\omega]$ is an *ordinal,* not quite to identify with *set theoretical ordinal* ω^ω, because classical ordinal addition on that ordinal ω^ω does not commute, e.g. classically $\omega + 1 \neq 1 + \omega = \omega$. As linear *orders* (with non-infinite descent) the two are identical.

As is well known, consistency provability and *soundness* of a theory are strongly tied together. We get in fact even

Theorem on πR-framed objective soundness of theory PR$\mathbb{X}$a :

- for a $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate $\chi = \chi(a) : A \to 2$ we have

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to 2.$$

- more general, for $\mathbf{PR}\mathbb{X}\mathbf{a}$-maps $f,\ g : A \to B$ we have

$$\pi\mathbf{R} \vdash \ulcorner f\urcorner \check{\doteq}_k \ulcorner g\urcorner \implies f(a) \doteq g(a).$$

[Same for set theory $\mathbf{T}$ taken as frame]

Proof of first assertion is a slight generalisation of proof of *framed Internal Consistency* above as follows—take predicate χ instead of false :

Use *termination-conditioned soundness,* assertion (iii) directly:

Evaluation-effective internal provabiliity of $\ulcorner\chi\urcorner$ within $\mathbf{PR}\mathbb{X}\mathbf{a}$—i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner\chi\urcorner$—*implies* $\chi(a), a \in A$ free :

$$\begin{aligned}\mathbf{PR}\mathbb{X}\mathbf{a},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \wedge\ c_d\ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) \doteq 0 \\ \implies \chi(a) : \mathbb{N}^2 \times A \to 2.\end{aligned}$$

Boolean free-variables calculus, tautology

$$[\alpha \wedge \beta \Rightarrow \gamma] = [\neg\,[\alpha \Rightarrow \gamma] \Rightarrow \neg\beta]$$

(test with $\beta = 0$ as well as with $\beta = 1$),

gives from this, still with k, m, a free:

$$\begin{aligned}\pi\mathbf{R} \vdash \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \Rightarrow \chi(a)\,] \\ \implies c_d\ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) > 0 : (A \times \mathbb{N}) \times \mathbb{N} \to 2.\end{aligned}$$

As before, we apply non-infinit scheme (π) to ev_d, in combination with—choice of—*overall "negative"* condition

$$\psi = \psi(k, a) := \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \Rightarrow \chi(a)\,] : \mathbb{N} \times A \to 2,$$

and get—scheme (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to 2.$$

q. e. d. for first assertion.

For **proof** of second assertion, take in the above

$$\chi = \chi(a) := [\,f(a) \doteq g(a)\,] : A \to B^2 \to 2$$

and get

$$\begin{aligned}
&\pi\mathbf{R} \vdash \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \\
&\qquad\qquad \implies \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(j(k), \ulcorner f \doteq g \urcorner) \\
&\qquad\qquad\quad \text{(substitutivity into } \doteq\text{)} \\
&\qquad\qquad \implies \ [\, f(a) \doteq g(a) \,] : \mathbb{N} \times A \to 2 \quad \textbf{q. e. d.}
\end{aligned}$$

7.3 πR decision

As the kernel of decision for p. r. predicate $\chi = \chi(a) : A \to 2$ *by* theory $\pi\mathbf{R}$ we introduce a (partially defined) μ-recursive *decision algorithm* $\nabla\chi = \nabla_{\mathrm{PR}}\chi : \mathbb{1} \rightharpoonup 2$ for (individual) χ. This decision algorithm is viewed as a map of theory $\pi\widehat{\mathbf{R}}$, of *partial* $\pi\mathbf{R}$ maps.

As a *partial* p. r. map it is given—see chapter 2—by three (PR) data:

- its index domain $D = D_{\nabla\chi}$, typically (and here): $D \subseteq \mathbb{N}$,
- its enumeration $d = d_{\nabla\chi} : D \to \mathbb{1}$ of its *defined arguments,* as well as
- its *rule* $\widehat{\nabla} = \widehat{\nabla}\chi : D \to 2$ mapping indices k, k' in D pointing to the same argument $d(k) \doteq d(k')$ in domain $\mathbb{1}$, to the same *value* $\widehat{\nabla}(k) \doteq \widehat{\nabla}(k')$.

Now **define** alleged decision algorithm by fixing its *graph*

$$\nabla\chi = \langle (d, \widehat{\nabla}) : D \to \mathbb{1} \times 2 \rangle : \mathbb{1} \rightharpoonup 2$$

as follows:

Enumeration *domain for defined arguments* is to be

$$D = D_{\nabla\chi} \ =_{\text{def}} \ \{k : \neg\,\chi\,\mathrm{ct}_A(k) \,\vee\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner)\} \subset \mathbb{N},$$

with $\mathrm{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count, A assumed pointed.

Defined arguments *enumeration* is here "simply"

$$d \ =_{\text{def}} \ \Pi : D \xrightarrow{\subseteq} \mathbb{N} \xrightarrow{\Pi} \mathbb{1}$$

—not a priori a retraction or empty—, and *rule* is taken

$$\widehat{\nabla}(k) = \widehat{\nabla}\chi(k) \ =_{\text{def}} \ \begin{cases} \text{false if } \neg\,\chi\,\mathrm{ct}_A(k), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \end{cases} : D \to \mathbb{2}.$$

$\widehat{\nabla} : D \to \mathbb{2}$ is in fact a well defined *rule* for *enumeration* $d : D \to \mathbb{N} \to \mathbb{1}$ of *defined argument(s)* since by (earlier) *framed logical soundness theorem*

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

whence disjointness of the alternative within $D = D_{\nabla\chi}$.

This taken together means intuitively within $\pi\mathbf{R}$—and formally within set theory $\mathbf{T}$:

$$\nabla(k) = \nabla\chi(k) = \begin{cases} \text{false if } \neg\,\chi\,\mathrm{ct}_A(k), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner), \\ \textit{undefined otherwise.} \end{cases}$$

We have the following complete—metamathematical—**case distinction** on $D \subset \mathbb{N}$:

- **1st case,** termination: D has at least one ("total") PR point $\mathbb{1} \to D \subseteq \mathbb{N}$, and hence

$$t = t_{\nabla\chi} \ =_{\text{by def}} \ \mu D = \min\ D : \mathbb{1} \to D$$

  is a (total) p. r. point.

  Subcases:

  - **1.1st,** negative (total) **subcase:**
    $\neg\,\chi\,\mathrm{ct}_A(t) = \text{true}.$
    [Then $\pi\mathbf{R} \vdash\ \nabla\chi = \text{false}.$]
  - **1.2nd,** positive (total) **subcase:**
    $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) = \text{true}.$
    [Then $\pi\mathbf{R} \vdash\ \nabla\chi = \text{true},$
    by $\pi\mathbf{R}$-framed objective soundness of $\mathbf{PR}\mathbb{X}$.]

    These two subcases are ***disjoint,*** disjoint here by $\pi\mathbf{R}$ framed soundness of theory $\mathbf{PR}\mathbb{X}$ which reads

$$\begin{aligned} \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) &: \\ \mathbb{N} \times A \to \mathbb{2},\ k \in \mathbb{N} \text{ free},\ &\text{and } a \in A \text{ free}, \end{aligned}$$

    here in particular—substitute $t : \mathbb{1} \to \mathbb{N}$ into k free:

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) \implies \chi(a) : A \to \mathbb{2},\ a \text{ free}.$$

    So furthermore, by this framed soundness, in present **subcase:**

$$\pi\mathbf{R} \vdash \chi(a) \wedge \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) : A \to \mathbb{2}.$$

- **2nd case,** derived non-termination:

  $\pi\mathbf{R} \vdash \ D = \emptyset_{\mathbb{N}} \equiv \{\mathbb{N} : \mathrm{false}_{\mathbb{N}}\} \subset \mathbb{N}$

  [then in particular $\pi\mathbf{R} \vdash \ \neg\,\chi = \mathrm{false}_A : A \to \mathbb{2}$,

  so $\pi\mathbf{R} \vdash \chi$ in this case],

  and

  $\pi\mathbf{R} \vdash \ \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) : \mathbb{N} \to \mathbb{2}$, k free;

- **3rd,** remaining, *ill* **case** is:

  D (metamathematically) *has no (total) points* $\mathbb{1} \to D$, *but is nevertheless not empty.*

Take in the above the (disjoint) union of 2nd subcase of 1st case and of 2nd case, last assertion. And formalise last, remaining case frame $\pi\mathbf{R}$. Arrive at the following

Quasi-Decidability Theorem: p. r. predicates $\chi : A \to \mathbb{2}$ give rise within theory $\pi\mathbf{R}$ to the following **complete (metamathematical) case distinction:**

(a) $\pi\mathbf{R} \vdash \chi : A \to \mathbb{2}$ or else

(b) $\pi\mathbf{R} \vdash \neg\,\chi\,\mathrm{ct}_A\,t : \mathbb{1} \to D_{\nabla\chi} \to \mathbb{2}$

(*defined counterexample*), or else

(c) $D = D_{\nabla\chi}$ *non-empty, pointless,* formally: in this case we would have within $\pi\mathbf{R}$:

$[D \mathbin{\widehat{\circ}} \mu D \mathrel{\widehat{=}} \mathrm{true} : \mathbb{1} \rightharpoonup \mathbb{N} \to \mathbb{2}]$

and "nevertheless" for each p. r. point $p : \mathbb{1} \to \mathbb{N}$

$\neg\, D \circ p = \mathrm{true} : \mathbb{1} \to \mathbb{N} \to \mathbb{2}.$

We rule out the latter—general—possibility of a *non-empty, pointless* predicate, for quantified arithmetical frame theory $\mathbf{T}$ by gödelian ***assumption*** of ω-consistency which rules out above instance of ω-*inconsistency.*

For frame $\pi\mathbf{R}$ we rule it out by (corresponding) metamathematical ***assumption*** of "μ-consistency," as follows:

Intermission on two variants of ω-consistency:

Gödelian ***assumption*** of ω-consistency—non-ω-inconsistency—for a *quantified* arithmetical theory $\mathbf{T}$ reads:

For no p. r. predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\mathbf{T} \vdash \ (\exists\, n \in \mathbb{N})\, \varphi(n)$$

and (nevertheless)

$$\mathbf{T} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \neg\, \varphi(2),\ \ldots$$

Adaptation to (categorical) recursive theory $\pi\mathbf{R}$ is the following ***assumption*** of μ-consistency, non-μ-inconsistency for $\pi\mathbf{R}$:

For no p. r. predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\pi\mathbf{R} \vdash \ \varphi(\mu\varphi) \ =_{\text{by def}} \ \varphi \,\widehat{\circ}\, \mu\varphi \,\widehat{=}\, \text{true} : \mathbb{1} \rightharpoonup \mathbb{2}$$

and

$$\pi\mathbf{R} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \ldots\ ,\ \neg\, \varphi(\text{num}(\underline{n})),\ \ldots$$

For quantified $\mathbf{T}$ first line reads: $\mathbf{T} \vdash \exists n\, \varphi(n)$, and hence μ-consistency is equivalent to gödelian ω-consistency for such $\mathbf{T}$.

Alternative to μ-consistency: π-consistency.

By assertion (iii) of **Structure theorem** in chapter 2—*section lemma*—for theories $\widehat{\mathbf{S}}$ of partial p. r. maps, first factor $\mu\varphi : \mathbb{1} \rightharpoonup \mathbb{N}$ of (total) p. r. map true : $\mathbb{1} \to \mathbb{2}$ above is necessarily itself a—*totally defined*—PR map: Intuitively, a first factor of a total map cannot have undefined arguments, since these would be undefined for the composition.

Now consider—here available—(external) point evaluation into numerals[3], externalisation of objective evaluation

$$ev : \lceil \mathbb{1}, \mathbb{N} \rceil \xrightarrow{\cong} \lceil \mathbb{1}, \mathbb{N} \rceil \times \mathbb{1} \xrightarrow{ev} \mathbb{N} \xrightarrow{\cong} \nu\mathbb{N} \subseteq \lceil \mathbb{1}, \mathbb{N} \rceil$$

of point codes into (internal) numerals, $ev(u) \doteq u \in \lceil \mathbb{1}, \mathbb{N} \rceil$.

This externalised evaluation $\underline{ev}$ is ***assumed***—meta-axiom of π-consistency—to (correctly) terminate:

$$\pi\mathbf{R}(\mathbb{1}, \mathbb{N}) \supset \mathrm{num}\, \underline{\mathbb{N}} \ni \underline{ev}(p) =^{\pi} p \in \pi\mathbf{R}(\mathbb{1}, \mathbb{N}).$$

Comment: π-consistency means *Semantical Completeness* of descent axiom (π), this axiom is modeled into the external world of p. r. Metamathematic. But π-consistency is somewhat stronger: it assumes termination of $\underline{ev}$ instead of non-infinite descent.

Non-μ-inconsistency (of $\pi\mathbf{R}$) is then a consequence of π-consistency of theory $\pi\mathbf{R}$ above:

$$\begin{aligned} &\pi\mathbf{R} \vdash \ \mathrm{true} = \varphi(\mu\varphi) = \varphi \,\widehat{\circ}\, \mu\varphi = \varphi \circ \mu\varphi : \mathbb{1} \to \mathbb{N} \to \mathbb{2} \\ &\text{entails } \pi\mathbf{R} \vdash \ \neg\, (\neg\, \varphi(\mathrm{num}(\underline{n}_0))), \text{with } \ \underline{ev}(\mu\varphi) = \mathrm{num}(\underline{n}_0). \end{aligned}$$

End of Intermission.

[3] Lassmann 1981

First **consequence:** Theory $\pi\mathbf{R}$ admits no non-empty predicative subset $\{n \in \mathbb{N} : \varphi(n)\} \subseteq \mathbb{N}$ such that for each numeral $\mathrm{num}(\underline{n}) : \mathbb{1} \to \mathbb{N}$

$$\pi\mathbf{R} \vdash \neg\,\varphi \circ \mathrm{num}(\underline{n}) : \mathbb{1} \to \mathbb{N} \to \mathbb{2}.$$

This rules out—in *quasi-decidability* above—possibility (c) for decision domain $D = D_{\nabla_\chi} \subseteq \mathbb{N}$ of decision operator ∇_χ for predicate $\chi : A \to \mathbb{2}$, and we get two unexpected results:

Decidability theorem: Each free-variable p. r. predicate $\chi : A \to \mathbb{2}$ gives rise to the following **complete case distinction** within, by $\pi\mathbf{R}$:

- Under ***assumption*** of μ-consistency or π-consistency for $\pi\mathbf{R}$:
  - $\pi\mathbf{R} \vdash\ \chi(a) : A \to \mathbb{2}$ (*theorem*) or
  - $\pi\mathbf{R} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla\chi} \to \mathbb{2}$ (*defined counterexample.*)
- Under ***assumption*** of ω-consistency for set theory $\mathbf{T}$:
  - $\mathbf{T} \vdash\ \chi(a) : A \to \mathbb{2}$ (*theorem*) or
  - $\mathbf{T} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla\chi} \to \mathbb{2}$, i. e. $\mathbf{T} \vdash (\exists\, a \in A)\,\neg\,\chi(a)$.

Take here, in case of set theory $\mathbf{T}$, for predicate χ, $\mathbf{T}$'s own free-variable consistency formula $\mathrm{Con}_{\mathbf{T}} = \neg\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbb{2}$, and get, under ***assumption*** of ω-consistency for $\mathbf{T}$, **consistency decidability** for $\mathbf{T}$.

This contradiction to (the postcedent) of Gödel's **2nd Incompleteness theorem** shows that the ***assumption*** of ω-consistency for set theories $\mathbf{T}$ must fail.

Now take in the theorem for χ $\pi\mathbf{R}$'s own free variable p. r. consistency formula

$$\mathrm{Con}_{\pi\mathbf{R}} = \neg\,\mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to \mathbb{2} \text{ and get}$$

Consistency Decidability for descent theory $\pi\mathbf{R}$:

- $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}} : \mathbb{1} \to \mathbb{2}$ or else
- $\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_{\pi\mathbf{R}}$, will say

  $\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(\mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \mathrm{false} \urcorner), \ulcorner \mathrm{false} \urcorner) = \mathrm{true}$ **q. e. d.**

Consistency provability theorem: $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$, under ***assumption*** of π-consistency of theory $\pi\mathbf{R}$.

Proof: Suppose we have 2nd alternative in *consistency decidability* above,

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(t, \ulcorner \mathrm{false} \urcorner),$$

$t\ =_{\mathrm{def}}\ \mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{1} \to \mathbb{N}$, necessarily ("total") PR. Meta p. r. point evaluation $\underline{ev}$ would turn—π-consistency—t into a numeral $\mathrm{num}(\underline{k}_0) : \mathbb{1} \to \mathbb{N}$, $\underline{k}_0 \in \underline{\mathbb{N}}$, $\mathrm{num}(\underline{k}_0) =^{\pi} t$, hence

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}_0), \ulcorner \mathrm{false} \urcorner).$$

But by derivation-into-*proof* internalisation we have

$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}), \ulcorner \chi \urcorner)$ (only) iff $\pi\mathbf{R} \vdash_{\underline{k}} \chi$, whence we would get inconsistency $\pi\mathbf{R} \vdash_{\underline{k}_0}$ false, (and an inconsistent theory derives everything.)

This rules out in fact 2nd alternative in consistency decidability and so proves the **theorem,** here our main **goal.**

For **proof** of *soundness* of $\pi\mathbf{R}$ below we need

ν**-Lemma** for theory $\pi\mathbf{R}$:

(i) family $\nu_A : A \to [\mathbb{1}, A]_\pi = \lceil \mathbb{1}, A \rceil / \check{=}^\pi$ is a natural transformation, will say

$$\begin{aligned} &(\nu_B \circ f)(a) = \nu_B(f(a)) \\ &\check{=}^\pi_{k(a)} \ulcorner f \urcorner \odot \nu_A(a) \qquad\qquad (*) \\ &= [\mathbb{1}, f]_\pi(\nu_A(a)), \\ &k(a) : A \to \mathbb{N} \text{ suitable PR.} \end{aligned}$$

As a commuting DIAGRAM:

$$\begin{tikzcd} A \ni a \arrow[r, mapsto, "\nu_A"] \arrow[dd, mapsto, "f"] & \nu_A(a) \in \lceil \mathbb{1}, A \rceil \arrow[d, mapsto, "\lceil \mathbb{1}, f \rceil"] \\ & \ulcorner f \urcorner \odot \nu_A(a) \arrow[d, phantom, "\check{=}^\pi"] \\ B \ni f(a) \arrow[r, mapsto, "\nu_B"] & \nu_B f(a) \in \lceil \mathbb{1}, B \rceil \end{tikzcd}$$

(ii) $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$ is injective, i. e.

$$\nu(m) \check{=}^\pi \nu(n) \implies m \doteq n.$$

(iii) same for all objects A of $\pi\mathbf{R}$: $\nu_A = \nu_A(a) : A \to [\mathbb{1}, A]_\pi$ is injective.

Proof: We show assertion (i) by structural recursion on $f : A \to B$.

anchor cases $f = \mathrm{id}_A$ as well as $f = 0 : \mathbb{1} \to \mathbb{N}$ are obvious.
anchor case $f = s : \mathbb{N} \to \mathbb{N}$:
$\nu(s(a)) \ =_{\text{by def}} \ulcorner s \urcorner \odot \nu(a) = [\mathbb{1}, s]\,(\nu(a))$.

Map composition $g \circ f : A \to B \to C$: combine the two commuting squares for f and for g into commuting rectangle for $g \circ f$.

cartesian Structure: use

$$\nu_{(A \times B)} \ =_{\text{by def}} \ \mathrm{ind} \circ (\nu_A \times \nu_B) :$$
$$A \times B \to \lceil \mathbb{1}, A \rceil \times \lceil \mathbb{1}, B \rceil \xrightarrow{\cong} \lceil \mathbb{1}, A \times B \rceil \to [\mathbb{1}, A \times B],$$

componentwise definition of (any) equality on cartesian product, as well as the universal properties of the cartesian product $A \times B$ and $[\mathbb{1}, A \times B] \cong [\mathbb{1}, A] \times [\mathbb{1}, B]$, projections $[\mathbb{1}, l], [\mathbb{1}, r]$.

Iterated $f^{\S}(a, n) : A \times \mathbb{N} \to A$ of (already tested) endo $f : A \to A$:

Straight forward by recursion on n, since iteration is repeated composition.

Assertion (ii) on injectivity of $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$:

$$\begin{aligned}
&\nu(m) \check{\doteq}^\pi \nu(n) \implies \ulcorner \doteq \urcorner \odot (\nu(m) \times \nu(n)) \check{\doteq}^\pi \ulcorner \mathrm{true} \urcorner \\
&\qquad \text{by internal substitutivity into predicative equality } \doteq \\
&\iff \lceil \mathbb{1}, \doteq \rceil \circ (\nu \times \nu)(m, n) \check{\doteq}^\pi \ulcorner \mathrm{true} \urcorner \\
&\implies \nu_{\mathbb{2}}[m \doteq n] \check{\doteq}^\pi \nu_{\mathbb{2}}(\mathrm{true}) \\
&\qquad \text{by naturality of transformation } \nu \\
&\implies m \doteq n, \text{ by } \textit{self-consistency} \text{ (!) of theory } \pi\mathbf{R}.
\end{aligned}$$

General ν injectivity assertion (iii) now follows from that special just above, from componentwise definition of ν—and componentwise definition of injectivity—on cartesian products (and restriction of both to predicative subobjects), via naturality of transformation $[\, \nu_A : A \to [\mathbb{1}, A]_\pi \,]_{A \in \pi\mathbf{R}}$ **q. e. d.**

This is to give self-consistency $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ to be equivalent to

Objective soundness theorem for descent theory $\pi\mathbf{R}$:

- for $\pi\mathbf{R}$-maps $f,\ g : A \to B$:

$$\pi\mathbf{R} \vdash\ [\ulcorner f \urcorner \check{\doteq}^\pi_k \ulcorner g \urcorner] \implies f(a) \doteq_B g(a) : \mathbb{N} \times A \to \mathbb{2}.$$

- this gives in particular *logical soundness* of theory $\pi\mathbf{R}$:

  For a predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \chi \urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

  $a \in A$ free, meaning here $\forall a$, and $k \in \mathbb{N}$ free, meaning here $\exists k$.

Proof: Granted self-consistency of theory $\pi\mathbf{R}$ means just injectivity of numeralisation

$$\nu_{2} : 2 \to [1, 2]_\pi = \lceil 1, 2 \rceil / \check{=}^\pi.$$

The **Lemma** deduces that this injectivity carries over first to numeralisation $\nu_{\mathbb{N}} = \nu : \mathbb{N} \to [1, \mathbb{N}]_\pi$, and then to all numeralisations

$$\nu_B : B \to [1, B]_\pi, \; B \text{ a } \pi\mathbf{R} \text{ object.}$$

Now compatibility of internal composition with internal equality as well as—**Lemma** again—naturality of transformmation $\nu_A : A \to [1, A]_\pi$ give

$$\begin{aligned}
\pi\mathbf{R} \vdash [\ulcorner f \urcorner \check{=}^\pi_k \ulcorner g \urcorner] \\
&\implies \ulcorner f \urcorner \odot \nu_A(a) \check{=}^\pi \ulcorner g \urcorner \odot \nu_A(a) \\
&\implies \nu_B(f(a)) \check{=}^\pi \nu_B(g(a)) \\
&\implies f(a) \doteq g(a),
\end{aligned}$$

the latter implication following from injectivity of $\nu_B : B \to [1, B]_\pi$ **q. e. d.**

ω-**completeness theorem** for theory $\pi\mathbf{R}$: theory $\pi\mathbf{R}$ admits the following scheme of *test by all internal numerals:*

$$(\omega\text{−Comp}) \quad \frac{\begin{array}{l} \chi = \chi(a) : A \to 2 \text{ predicate,} \\ k = k(a) : A \to \mathbb{N} \text{ such that} \\ \pi\mathbf{R} \vdash \; \mathrm{Prov}_{\pi\mathbf{R}}(k(a), \ulcorner \chi \urcorner \odot \nu_A(a)) : A \to 2 \end{array}}{\pi\mathbf{R} \vdash \; \chi : A \to 2.}$$

Proof: By ν naturality—within $\pi\mathbf{R}$—the antecedent gives

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(k'(a), \nu_{\mathbb{2}} \circ \chi(a)) : A \to \mathbb{2},$$

and from this, by $\pi\mathbf{R}$ self-consistency: injectivity of $\nu_{\mathbb{2}}$ within $\pi\mathbf{R}$,

$$\pi\mathbf{R} \vdash \chi(a) : A \to \mathbb{2} \quad \textbf{q. e. d.}$$

Interpretation: The $\nu_A(a), a \in A$ are jointly epic, νA lies *dense* in $[\mathbb{1}, A]_\pi$. theory $\pi\mathbf{R}$ is in particular internally μ-consistent, object $\mathbb{1}$ is an internal separator, all of this with respect to $\pi\mathbf{R}$ maps (on object language level). Would it work for (free variable) internal map codes either?

Separator question: Can we then have/assume this test to work on the external level too? can we have/***assume*** consistently at least object $\mathbb{1}$ to be/to become a *separator* for category $\pi\mathbf{R}$?

Attempt to an answer: logic/arithmetic externalisation of axioms and theorems, as opposite to—successfull—internalisation/arithmetisation seems me to be legitimate/consistent: both internalisation and externalisation can be seen/formalised as preserving/reflecting logical *invariants.* A theory $\mathbf{T}$ for which this is not always possible—Consistency/*consistency provability*—has a defect in this regard, it is not *sound* in the technical sense, see Smorynski 1977.

Conclusion: descent theory $\pi\mathbf{R}$—in the role of metamathematic—derives its own *consistency* (formula) as well as—see below—the *inconsistency* (formulae) for set theories $\mathbf{T}$, the latter including Peano-arithmetic $\mathbf{PA}^+$ with order of $\mathbb{N}[\omega]$ to satisfy finite descent.

All of this under ***assumption***, meta-axiom, that theory $\pi\mathbf{R}$ is π-consistent, that it externalises its axiom (π) into (correct) termination of (external) evaluation $\underline{ev}$.

The $\pi\mathbf{R}$ (in part) internal version of μ-consistency, consequence of π-consistency, is ω-completeness above.

Inconsistency question: Are quantified arithmetical theories $\mathbf{T}$, in particular theory $\mathbf{PA}$, even inconsistent?

By Gödel's 2nd Incompleteness theorem, first assertion, $\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}}$ if $\mathbf{T}$ consistent, hence $\pi\mathbf{R} \nvdash \mathrm{Con}_{\mathbf{T}}$ if $\mathbf{T}$ consistent: this since $\mathbf{T}$ is an extension of $\pi\mathbf{R}$. But then, by Decidability theorem above, for $\pi\mathbf{R}$ and p. r. free-variable predicate $\mathrm{Con}_{\mathbf{T}} = \neg\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to 2$,

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_{\mathbf{T}}, \ \ [\text{a fortiori } \ \mathbf{T} \vdash \neg\,\mathrm{Con}_{\mathbf{T}}.]$$

Now if we take as metamathematic the external version $\underline{\mathbf{PR}}$ of fundamental theory $\mathbf{PR}$, then the consistency questions are open.

But if we take as metamathematic an external version $\underline{\pi\mathbf{R}}$ of descent theory $\pi\mathbf{R}$, then we get in fact consistency—***constructive consistency***—of p. r. theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}\mathbf{a}$, and of descent theory $\pi\mathbf{R}$, as well as inconsistency of set theories $\mathbf{T}$.

Problems:

(1) Is axiom scheme (π) redundant, $\pi\mathbf{R} \cong \mathbf{PR}\mathbb{X}\mathbf{a}$? Certainly not, since isotonic maps from reverse-lexicographically ordered $\mathbb{N} \times \mathbb{N}, \ldots, \mathbb{N}^+ \equiv \mathbb{N}[\omega] \equiv \omega^\omega$ to $\mathbb{N}$ are not available.

(2) Can we get *internal* soundness for theory $\pi\mathbf{R}$ itself? Up to now we have only *Objective* soundness: this is the one considered by

mathematical logicians. Internal soundness (of *evaluation* versus the object language level) is a challenging open Problem with present approach.

(3) Can our metamathematical assumptions for descent theory $\pi\mathbf{R}$, $\mathbb{1}$ a sepaerator object and $\pi\mathbf{R}$ μ-consistent, be *formalised* as theorems of $\pi\mathbf{R}$? For the former, ω-completeness does the job. For the latter: μ-consistency internalises via evaluation of internal points into internal numerals.

(4) Can we ***assume*** consistently even that object $\mathbb{1}$ is a *generator* for category $\pi\mathbf{R}$, i. e. that a p. r. map $F : \pi\mathbf{R}(\mathbb{1}, A) \to \pi\mathbf{R}(\mathbb{1}, B)$ given generates a $\pi\mathbf{R}$ map $f : A \to B$ such that

$$F = \pi\mathbf{R}(\mathbb{1}, f) : \pi\mathbf{R}(\mathbb{1}, A) \to \pi\mathbf{R}(\mathbb{1}, B)?$$

This holds for **set** theory. We make this ***assumption*** for theory $\pi\mathbf{R}$. This is my answer to a question to Erich Kähler in 1963:

Aber Sie benutzen doch schon natuerliche Zahlen zur Beschreibung der Mengenlehre, mit der Sie die natürlichen Zahlen begründen wollen?

Kähler's answer: *diese Frage werden Sie später beantwortet bekommen.*

Discussion

- The claim for **set** theories $\mathbf{T}$ is that $\mathbf{T}$ derives $\neg\,\mathrm{Con}_{\mathbf{T}}$ which formally denies Gödel's second incompleteness theorem: its second postcedent and hence the ***assumption*** of ω-consistency for

PM, **ZF**, and **NGB**. Gödel himself has been said to be not completely convinced of this assumption.

- All set theories considered here are standard recursively axiomatized extensions of primitive recursive arithmetic **PR**. Everybody then expects for these set theories **T** ω-consistency. But this is only an *assumption.* Remains the possibility that present text contains a formal irreparable error. If so, where?

- [4] Theory **PA** is *not* formally concerned by present inconsistency argument since *descent* scheme (π) needed for its proof in **set** theory *nested* induction, available only(?) in higher order framework, another germ of inconsistency, cf. RCF 3 in the references.

- Axiomatisation and predicate $\mathrm{Prov}_{\mathbf{T}}$ of "being a *proof* for", are constructed in categorical parallel to Smorynski, and to Gödels predicate 45. $x\,B\,y$, x ist ein *Beweis* für die *Formel* y; not to Rosser's $\mathrm{Prov}^{R}_{\mathbf{T}}$.

- The referee for **TAC** (*Theory and Applications of Categories*) to a previous version Pfender 2012/13 of present work, critisized *poor English writing* and that *the author seems not to be aware of the developments of Categorical Logic in the past decades.* That is true, I do not need them. For me relevant development of category theory found its end with the 1988 category conference at Louvain-la-Neuve celebrating Sammy Eilenberg's 75th birthday. This referee did not discuss results and proofs.

[4] added in consequence of J. Wleczyk's comment

- Recently, the former chief of *Zentralblatt Math* engaged himself to find a reviewer. The reactions were *no answer* or *not understood.*

- I publish a very short—pp. 16—extract *Consistency Decision* in the **arXiv.** It considers just **set** theory as frame. It does not need the full categorical combinatorics of the present approach. It is self-contained, logically independent of present work.

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Bibliography

[1] J. Barwise ed. 1977: *Handbook of Mathematical Logic.* North Holland.

[2] H.-B. Brinkmann, D. Puppe 1969: *Abelsche und exakte Kategorien, Korrespondenzen.* Lecture Notes in Math. **96.**

[3] L. Budach, H.-J. Hoehnke 1975: *Automaten und Funktoren.* Akademie-Verlag Berlin.

[4] C. Chevalley 1956: *Fundamental Concepts of Algebra.* Academic Press.

[5] H. Ehrig, W. Kühnel, M. Pfender 1975: Diagram Characterization of Recursion. LN in Comp. Sc. **25,** 137-143.

[6] H. Ehrig, M. Pfender und Studenten 1972: *Kategorien und Automaten.* De Gruyter.

[7] S. Eilenberg, C. C. Elgot 1970: *Recursiveness.* Academic Press.

[8] S. Eilenberg, G. M. Kelly 1966: Closed Categories. *Proc. Conf. on Categorical Algebra*, La Jolla 1965, pp. 421-562. Springer.

[9] S. Eilenberg, S. Mac Lane 1945: General Theory of natural Equivalences. *Trans. AMS* **58,** 231-294.

[10] U. Felgner 2012: Das Induktions-Prinzip. *Jahresber. DMV* **114,** 23-45.

[11] M. P. Fourman 1977: The Logic of Topoi. Part D.6 in Barwise ed. 1977. *Handbook of Mathematical Logic*. North Holland.

[12] G. Frege 1879: *Begriffsschrift.* Reprint in "Begriffsschrift und andere Aufsätze", 2te Auflage 1971, I. Angelelli editor. Georg Olms Verlag Hildesheim, New York.

[13] P. J. Freyd 1972: Aspects of Topoi. *Bull. Australian Math. Soc.* **7,** 1-76.

[14] K. Gödel 1931: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatsh. der Mathematik und Physik* **38,** 173-198.

[15] R. L. Goodstein 1957/64: *Recursive Number Theory.* A Development of Recursive Arithmetic in a Logic-Free Equation Calculus. North-Holland.

[16] R. L. Goodstein 1971: *Development of Mathematical Logic,* ch. 7: Free-Variable Arithmetics. Logos Press.

[17] P. T. Johnstone 1977: *Topos Theory.* Academic Press

[18] A. Joyal 1973: Arithmetical Universes. Talk at Oberwolfach.

[19] W. Kühnel, M. Pfender, J. Meseguer, I. Sols 1977: Primitive recursive algebraic theories and program schemes. *Bull. Austral. Math. Soc.* **17,** 207-233.

[20] J. Lambek, P. J. Scott 1986: *Introduction to higher order categorical logic.* Cambridge University Press.

[21] M. Lassmann 1981: Gödel's Nichtableitbarkeitstheoreme und Arithmetische Universen. Diploma Thesis. Techn. Univ. Berlin.

[22] F. W. Lawvere 1964: An Elementary Theory of the category of Sets. *Proc. Nat. Acad. Sc. USA* **51,** 1506-1510.

[23] F. W. Lawvere 1970: Quantifiers and Sheaves. *Actes du Congrès International des Mathématiciens, Nice,* Tome I, 329-334.

[24] F. W. Lawvere, S. H. Schanuel 1997, 2000: *Conceptual Mathematics.* Cambridge University Press.

[25] S. Mac Lane 1972: *Categories for the working mathematician.* Springer.

[26] M. E. Maietti 2010: Joyal's arithmetic universe as list-arithmetic pretopos. *Theory and Applications of Categories* **24**(3), 39-83.

[27] G. Osius 1974: Categorical set theory: a characterisation of the category of sets. *J. Pure and Appl. Alg.* **4**, 79-119.

[28] B. Pareigis 1969: *Kategorien und Funktoren.* Teubner.

[29] B. Pareigis 2004: *Category Theory.* pdf Script, author's Home page LMU München.

[30] R. Péter 1967: *Recursive Functions.* Academic Press.

[31] M. Pfender 1974: Universal Algebra in S-Monoidal Categories. *Algebra-Berichte Nr. 20,* Mathematisches Institut der Universität München. Verlag Uni-Druck München.

[32] M. Pfender 2008b: RCF 2: Evaluation and Consistency. arXiv:0809.3881v2 [math.CT]. Has a **gap.**

[33] M. Pfender 2008c: RCF 3: Map-Code Interpretation via Closure. arXiv:0809.4970v1 [math.CT].

[34] M. Pfender 2012: *Arithmetical Foundations,* α version,

http://www3.math.tu-berlin.de/preprints/files/Preprint-38-2012.pdf

[35] M. Pfender 2012/2013: Inconsistency Provability for Set Theory. Submission to **TAC,** rejected.

[36] M. Pfender 2013: *Arithmetical Foundations – Excerpt,* β version, arXiv [math.LO, cs.LO].

[37] M. Pfender, M. Kröplin, D. Pape 1994: Primitive Recursion, Equality, and a Universal Set. *Math. Struct. in Comp. Sc.* **4,** 295-313.

[38] M. Pfender, R. Reiter, M. Sartorius 1982: Constructive Arithmetics. *Lecture Notes in Math.* **962,** 228-236.

[39] R. Reiter 1980: Mengentheoretische Konstruktionen in arithmetischen Universen. Diploma Thesis. Techn. Univ. Berlin.

[40] R. Reiter 1982: Ein algebraisch-konstruktiver Abbildungskalkül zur Fundierung der elementaren Arithmetik. Dissertation, rejected by Math. dpt. of TU Berlin.

[41] L. Romàn 1989: Cartesian categories with natural numbers object. *J. Pure and Appl. Alg.* **58,** 267-278.

[42] M. Sartorius 1981: Kategorielle Arithmetik. Diploma Thesis. Techn. Univ. Berlin.

[43] Th. Skolem 1970: *Selected Works in Logic.* Universitetsforlaget Oslo - Bergen - Tromsö.

[44] C. Smorynski 1977: The Incompleteness Theorems. Part D.1 in Barwise ed. 1977.

[45] A. Tarski, S. Givant 1987: *A formalization of set theory without variables.* AMS Coll. Publ. vol. 41.

[46] A. Yashuhara 1971: *Recursive function theory and logic.* Academic Press.